Z-Transform

z-Transform

- The z-transform is the most general concept for the transformation of discrete-time series.

- The Laplace transform is the more general concept for the transformation of continuous time processes.

The Transforms

The Laplace transform of a function *f*(*t*):

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

The one-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The two-sided z-transform of a function *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Relationship to Fourier Transform

Note that expressing the complex variable *z* in polar form reveals the relationship to the Fourier transform:

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \; or$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-i\omega n}, \; and \;\; if \;\; r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}$$

which is the Fourier transform of *x*(*n*).

Region of Convergence

The z-transform of *x*(*n*) can be viewed as the Fourier transform of *x*(*n*) multiplied by an exponential sequence r^{-n}, and the z-transform may converge even when the Fourier transform does not.

By redefining convergence, it is possible that the Fourier transform may converge when the z-transform does not.

For the Fourier transform to converge, the sequence must have finite energy, or:

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

Convergence, continued

The power series for the z-transform is called a Laurent series:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence, and therefore the z-transform and all its derivatives must be continuous functions of *z* inside the region of convergence.

In general, the Laurent series will converge in an annular region of the z-plane.

Some Special Functions

First we introduce the Dirac delta function (or unit sample function):

$$\delta(n)=\begin{cases}0, & n\neq 0\\ 1, & n=0\end{cases} \quad \textbf{or} \quad \delta(t)=\begin{cases}0, & t\neq 0\\ 1, & t=0\end{cases}$$

This allows an arbitrary sequence *x*(*n*) or continuous-time function *f*(*t*) to be expressed as:

$$x(n)=\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$f(t)=\int_{-\infty}^{\infty} f(x)\delta(x-t)dt$$

Convolution, Unit Step

These are referred to as discrete-time or continuous-time convolution, and are denoted by:

$$x(n) = x(n) * \delta(n)$$

$$f(t) = f(t) * \delta(t)$$

We also introduce the unit step function:

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note also:

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

Poles and Zeros

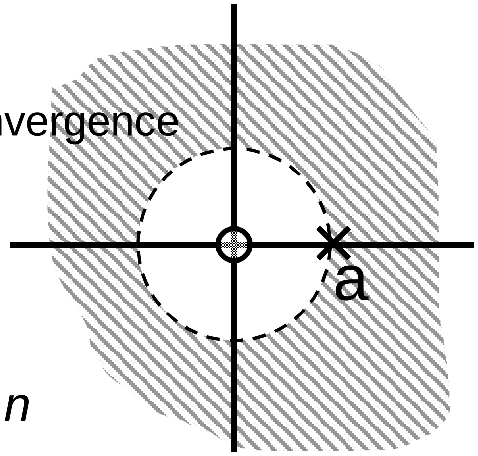
When $X(z)$ is a rational function, i.e., a ration of polynomials in z, then:

1. **The roots of the numerator polynomial are referred to as the zeros of $X(z)$, and**
2. **The roots of the denominator polynomial are referred to as the poles of $X(z)$.**

Note that no poles of $X(z)$ can occur within the region of convergence since the z-transform does not converge at a pole.

Furthermore, the region of convergence is bounded by poles.

Example



$$x(n) = a^n u(n)$$

The z-transform is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Which converges to:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad for \quad |z| > |a|$$

Clearly, *X*(*z*) has a zero at *z* = 0 and a pole at *z* = *a*.

Convergence of Finite Sequences

Suppose that only a finite number of sequence values are nonzero, so that:

$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

Where n_1 and n_2 are finite integers. Convergence requires

$$|x(n)| < \infty \ for \ n_1 \le n \le n_2.$$

So that finite-length sequences have a region of convergence that is at least $0 < |z| < \infty$, and may include either $z = 0$ or $z = \infty$.

Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z - transform.}$$

Properties

- z-transforms are linear:

$$\mathbf{Z}\left[ax(n)+by(n)\right]=aX(z)+bY(z)$$

- The transform of a shifted sequence:

$$\mathbf{Z}\left[x(n+n_0)\right]=z^{n_0}X(z)$$

- Multiplication:

$$\mathbf{Z}\left[a^n x(n)\right]=Z(a^{-1}z)$$

But multiplication will affect the region of convergence and all the pole-zero locations will be scaled by a factor of a.

Convolution of Sequences

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

Then

$$W(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k) \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k) z^{-n}$$

let $m = n - k$

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{m=-\infty}^{\infty} y(m) z^{-m} \right] z^{-k}$$

$W(z) = X(z)Y(z)$ for values of z inside the regions of convergence of both.

Properties of Z-Transform

- Derivative
  - If X(z) is the z-transform of x(n), the z-transform of is

$$nx(n) \stackrel{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

- Initial value theorem

  If X(z) is the z-transform of x(n) and x(n) is equal to zero for n<0, the initial value, x(0), maybe be found from X(z) as follows:

$$x(0) = \lim_{z\to\infty} X(z)$$

Table 4-2 Properties of the z-Transform

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0}X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$\|\alpha\|R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z\frac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y, respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.

More Definitions

Definition. Periodic. A sequence $x(n)$ is periodic with period λ if and only if $x(n) = x(n + \lambda)$ for all n.

Definition. Shift invariant or time-invariant. Consider a sequence $y(n)$ as the result of a transformation T of $x(n)$. Another interpretation is that T is a system that responds to an input or stimulus $x(n)$:

$$y(n) = T[x(n)].$$

The transformation T is said to be shift-invariant or time-invariant if:

$$y(n) = T\,[x(n)] \text{ implies that } y(n - k) = T\,[x(n - k)]$$

For all k. “Shift invariant” is the same thing as “time invariant” when n is time (t).

Let $h_k(n)$ be the response of the system to $\delta(n-k)$, a "spike" or shock occurring at $n = k$. Then:

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)T\left[\delta(n-k)\right]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n).$$

If we have time invariance of the transform T, then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n).$$

This implies that the system can be completely characterized by its impulse response $h(n)$. This obviously hinges on the stationarity of the series.

Definition. Stable System. A system is stable if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Which means that a bounded input will not yield an unbounded output.

Definition. Causal System. A causal system is one in which changes in output do not precede changes in input. In other words,

$$\text{If } x_1(n) = x_2(n) \text{ for } n \le n_0$$
$$\text{then } T[x_1(n)] = T[x_2(n)] \text{ for } n < n_0.$$

Linear, shift-invariant systems are causal iff $h(n) = 0$ for $n < 0$.

Given $y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$ let $x(n)$ be sinusoidal. That is,

let $x(n) = e^{i\omega n}$ for $-\infty < n < \infty$. Then

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega(n-k)} = e^{i\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}$$

Let $H(e^{i\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}$ so that

$$y(n) = H(e^{i\omega})e^{i\omega n}.$$

Here $H(e^{i\omega})$ is called the frequency response of the system whose impulse response is $h(n)$. Note that $H(e^{i\omega})$ is the Fourier transform of $h(n)$.

We can generalize this state that:

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega n}d\omega$$

These are the Fourier transform pair.

If $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$, then the transform is absolutely convergent and converges uniformly to a continuous function of ω.

This implies that the frequency response of a stable system always converges, and the Fourier transform exists.

If $x(n)$ is constructed from some continuous function $x_C(t)$ by sampling at regular periods T (called "the sampling period"), then $x(n) = x_C(nT)$ and $1/T$ is called the *sampling frequency* or *sampling rate.*

If ω_0 is the highest radial frequency of sinusoids comprising $x(nT)$, then

$$\omega_0 < \frac{2\pi}{T} \text{ or } \frac{1}{T} > \frac{w_0}{2\pi}$$

Is the sampling rate required to guarantee that $x_C(nT)$ can be used to fully recover $x_C(t)$, This sampling rate ω_0 is called the Nyquist rate (or frequency). Sampling at less than this rate will involve losing information from the time series.

Assume that the sampling rate is at least the Nyquist rate.

$$X(e^{i\omega T}) = \frac{1}{T} X_c(i\omega), \quad \frac{-\pi}{T} \le \omega \le \frac{\pi}{T}$$

From the continuous time Fourier transform :

$$x_c(t) = \frac{1}{2\pi} \int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} X_c(i\omega) e^{i\omega t} d\omega.$$

Combining :

$$x_c(t) = \frac{1}{2\pi} \int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} T X(e^{i\omega T}) e^{i\omega t} d\omega.$$

Since $X(e^{i\omega T}) = \sum_{k=-\infty}^{\infty} x_c(kT) e^{-i\omega Tk}$, we have

$$x_c(t) = \frac{T}{2\pi} \int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \left[\sum_{k=-\infty}^{\infty} x_c(kT) e^{-i\omega Tk} \right] e^{i\omega t} d\omega$$

Changing the order of summation and integration,

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_c(kT)\left[\frac{T}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} e^{i\omega(t-kT)}d\omega\right]$$

Evaluating the integral :

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_c(kt)\frac{\sin\left(\pi/T\right)(t-kT))}{\left(\pi/T\right)(t-kT)}.$$

NOTE: This equation allows for recovering the continuous time series from its samples. This is valid only for bandlimited functions.

Table 4-1 Common z-Transform Pairs

Sequence	z-Transform	Region of Convergence
$\delta(n)$	1	all z
$\alpha^n u(n)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n u(-n-1)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n u(n)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$\cos(n\omega_0)u(n)$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(n\omega_0)u(n)$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$