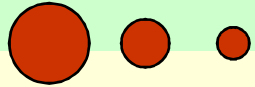
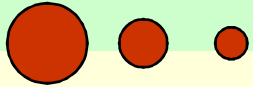


**Lecture :**  
**Some applications of z-transform**



- Quick review of z-transform
- Rational z-transform: poles, zeros, system function
- Time-domain behavior of LT systems. Stability and causality
- Inverse z-transform using partial fraction expansion

# Z-transform



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

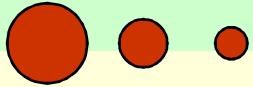
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) dz$$

- Relation to Laplace transform:

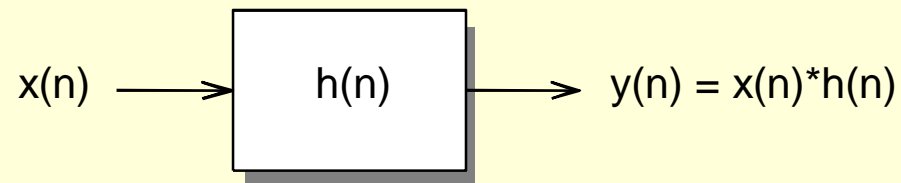
$$z = e^{s\Delta t}$$

- Relation to discrete time Fourier transform:  $z = 1 \cdot e^{j\omega}$

# Rational Z-transform



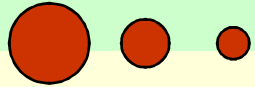
- Z-transform used to characterize systems.
- Convolution property:



$$y(n) = x(n) * h(n) \Rightarrow Y(z) = X(z)H(z)$$

- $H(z)$  – system function

# Rational z-transform: System function



- LCCDE system: 
$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$
- 

- Using linearity and time shift properties:

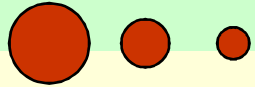
$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

⇓

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

# Poles and zeros

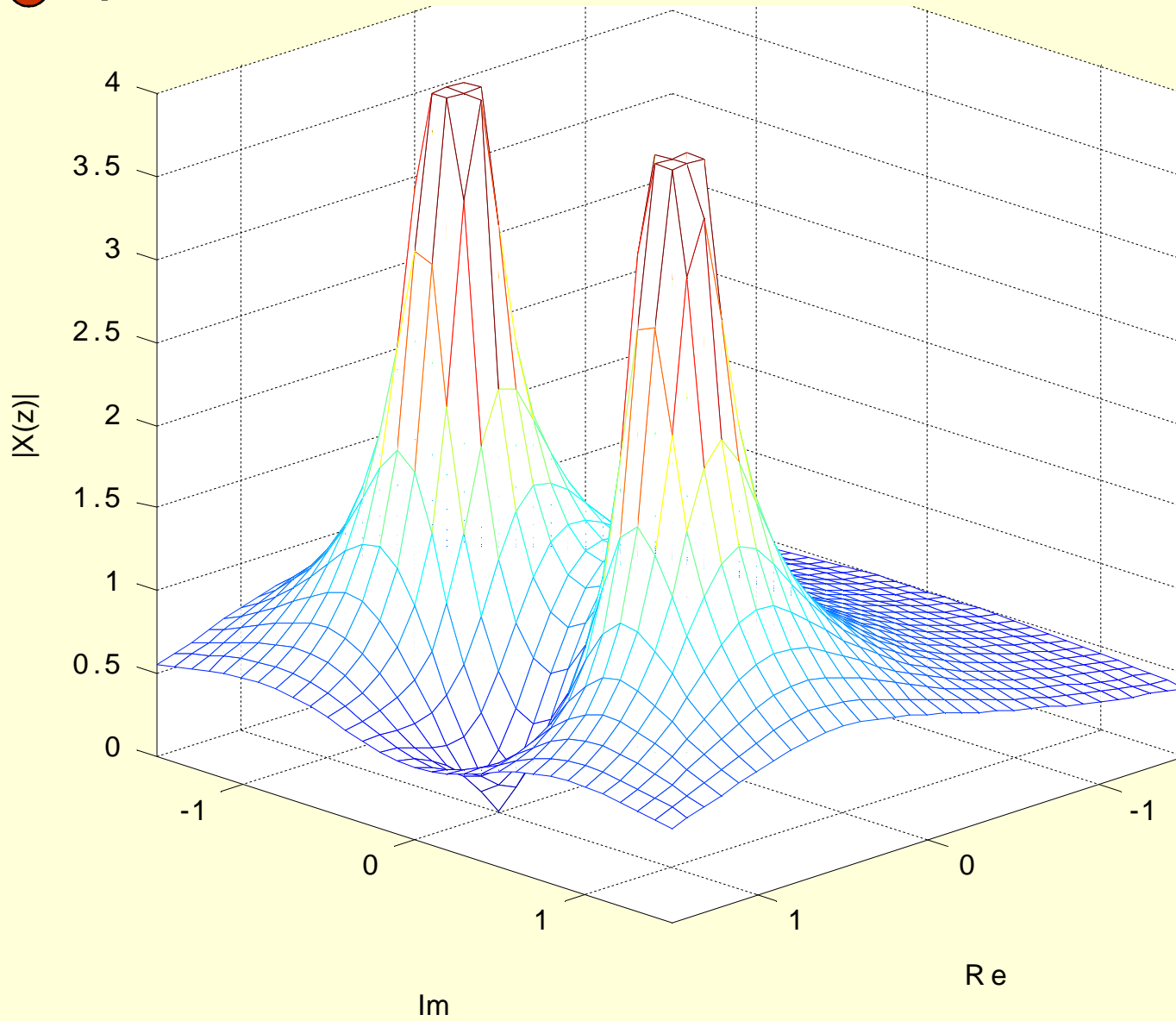
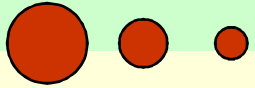


$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

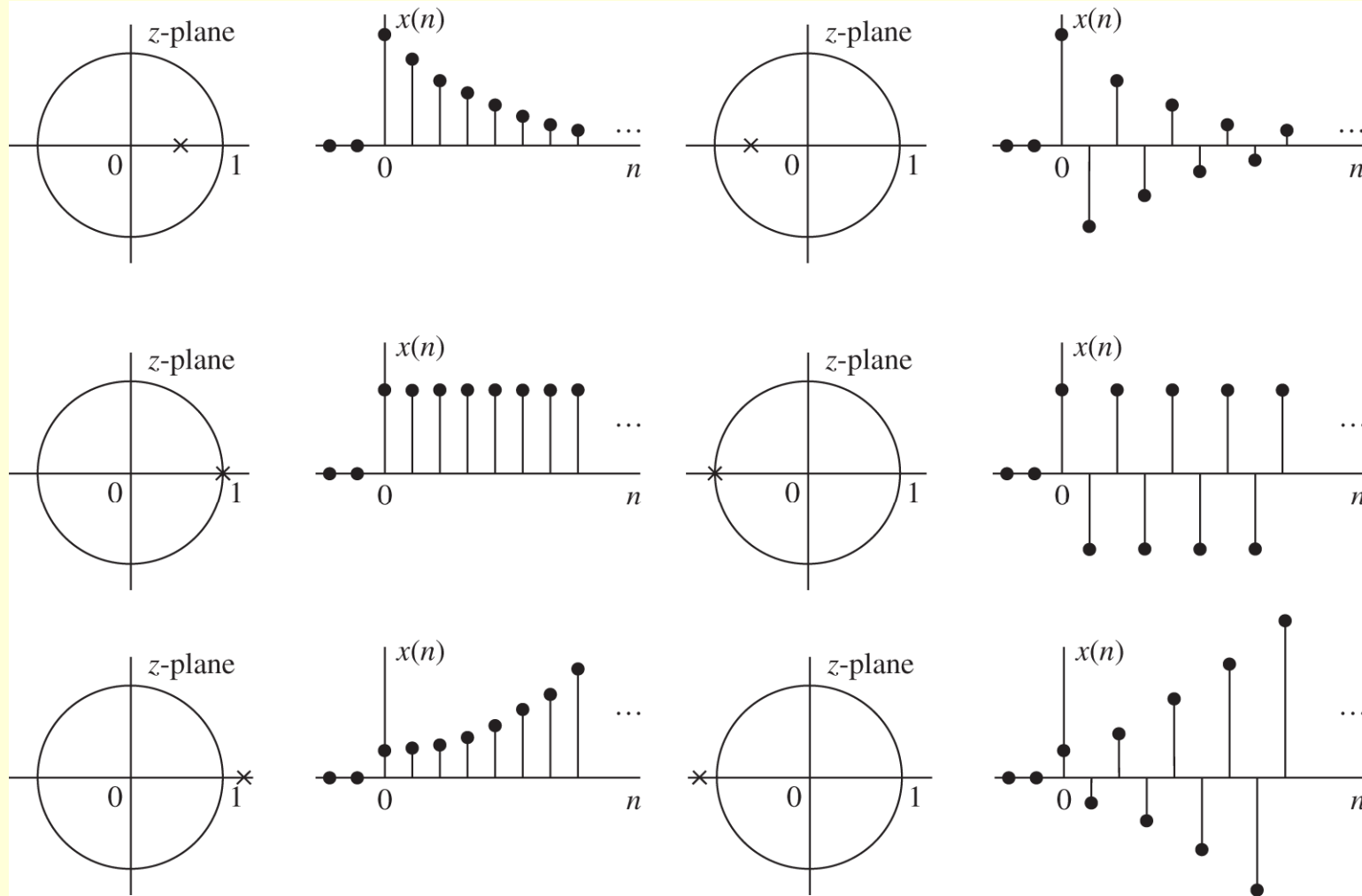
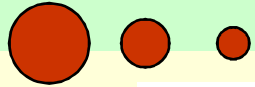
$$X(z) = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \dots (z - z_M)}{(z - p_1)(z - p_2) \dots (z - p_N)}$$

- The transform has  $M$  finite zeros at  $z = z_1, \dots, z_M$ ,  $N$  infinite poles at  $z = p_1, \dots, p_N$  and  $|N-M|$  zeros (if  $N > M$ ) or poles (if  $N < M$ ) at the origin  $z = 0$ .

# Pole-zero plots



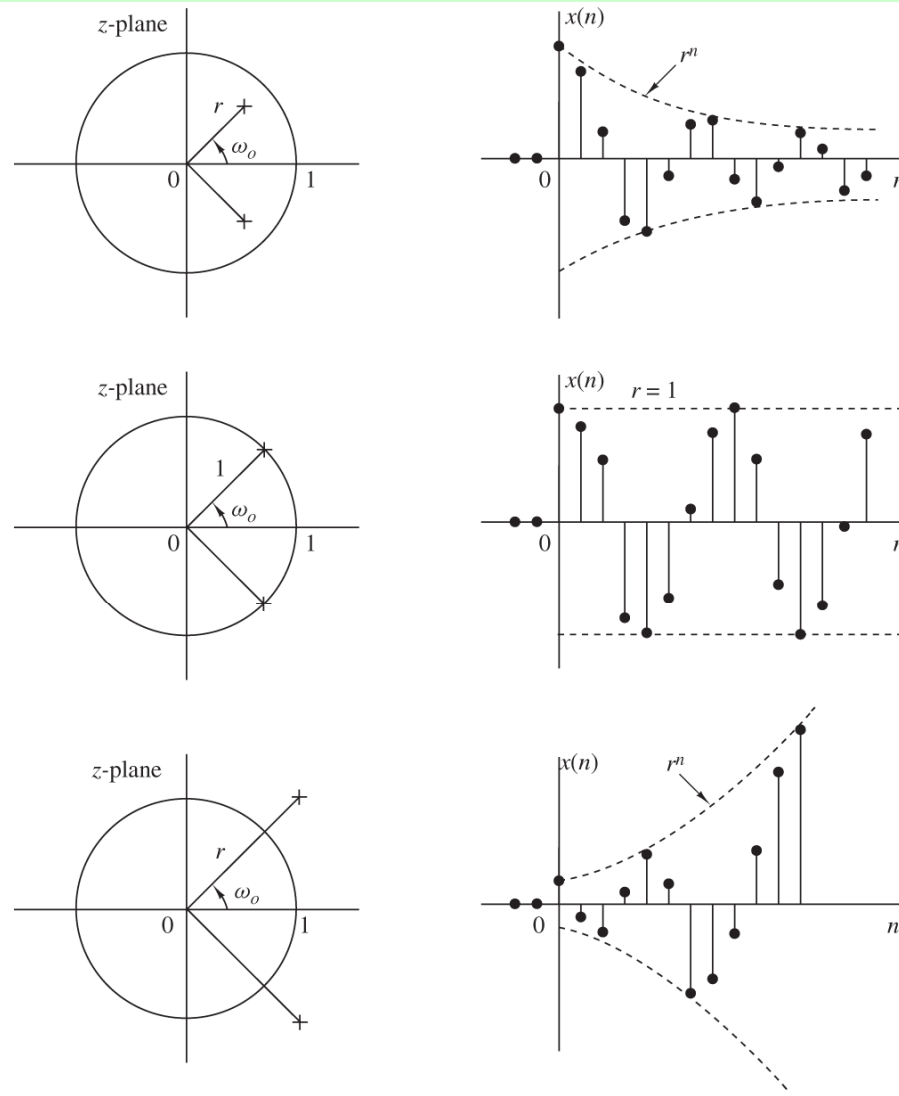
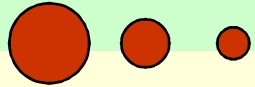
# Time-domain behavior of systems



**Figure 3.3.5** Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

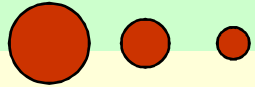


# Time domain behavior



**Figure 3.3.7** A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

# Inverse Z-tr.: partial fraction expansion



- Distinct poles:

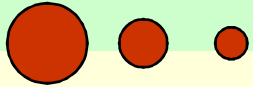
$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \dots + \frac{A_N}{z - p_N}$$

$$A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}$$

---

$$Z^{-1} \left\{ \frac{1}{1 - p_k z^{-1}} \right\} = \begin{cases} (p_k)^n u(n), & \text{if ROC: } |z| > |p_k| \\ -(p_k)^n u(-n - 1), & \text{if ROC: } |z| < |p_k| \end{cases}$$

## Example with multiple poles



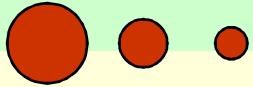
$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2} = \frac{A_1}{z + 1} + \frac{A_2}{z - 1} + \frac{A_3}{(z - 1)^2}$$

$$A_1 = \left. \frac{(z + 1)X(z)}{z} \right|_{z=-1} = \frac{1}{4}$$

$$A_3 = \left. \frac{(z - 1)^2 X(z)}{z} \right|_{z=1} = \frac{1}{2}$$

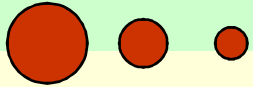
$$A_2 = \frac{d}{dz} \left[ \frac{(z - 1)^2 X(z)}{z} \right]_{z=1} = \frac{3}{4}$$

# Stability



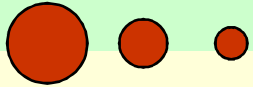
- A linear time-invariant system is BIBO stable if and only if the region of convergence of the system function includes the unit circle.
- A *causal* system is stable if all poles are inside the unit circle

## ***Pole – zero cancellation***



- Pole-zero cancellation occurs either in the system function itself, or in the product of the system function with the z-transform of input system.
- This is used to design filters.

# Summary



- Z-transform useful in system analysis
- Can be used to determine Fourier transform
- Used to decide if a system is causal and stable
- Used to determine the response of a system
- Used to design filters through pole-zero cancellation.