Lecture -3
Fourier Transform Properties

Fourier Transform Properties and Examples

Objectives:

1. Properties of a Fourier transform
   - **Linearity** & **time shifts**
   - **Differentiation**
   - **Convolution** in the frequency domain

While the **Fourier series/transform** is very important for representing a signal in the **frequency domain**, it is also important for **calculating a system's response (**convolution).

- A **system's transfer function** is the **Fourier transform** of its **impulse response**

- Fourier transform of a signal's **derivative** is **multiplication** in the **frequency domain**: $jwX(jw)$

- Convolution in the time domain is given by **multiplication** in the **frequency domain** (similar idea to log transformations)

Review: Fourier Transform

A CT signal *x*(*t*) and its frequency domain, Fourier transform signal, *X*(*jw*), are related by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$ *analysis*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$ *synthesis*

This is denoted by:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

For example:

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

Often you have tables for common Fourier transforms

The Fourier transform, *X*(*jw*), represents the **frequency content** of *x*(*t*).

It exists either when *x*(*t*)->0 as |*t*|->∞ or when *x*(*t*) is periodic (it generalizes the Fourier series)

Linearity of the Fourier Transform

The Fourier transform is a **linear function** of *x*(*t*)

$$x_1(t) \overset{F}{\leftrightarrow} X_1(j\omega)$$

$$x_2(t) \overset{F}{\leftrightarrow} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \overset{F}{\leftrightarrow} aX_1(j\omega) + bX_2(j\omega)$$

This follows directly from the definition of the Fourier transform (as the integral operator is linear) & it easily extends to an arbitrary number of signals

Like impulses/convolution, if we know the Fourier transform of simple signals, we can calculate the Fourier transform of more complex signals which are a linear combination of the simple signals

Fourier Transform of a Time Shifted Signal

We'll show that a Fourier transform of a signal which has a **simple time shift** is:

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

i.e. the original Fourier transform but **shifted in phase** by $-wt_0$

Proof

Consider the Fourier transform synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right)e^{j\omega t}d\omega$$

but this is the synthesis equation for the Fourier transform $e^{-jw_0 t}X(jw)$

Example: Linearity & Time Shift

Consider the signal (linear sum of two time shifted rectangular pulses)

$$x(t) = 0.5x_1(t-2.5) + x_2(t-2.5)$$

where $x_1(t)$ is of width 1, $x_2(t)$ is of width 3, centred on zero (see figures)

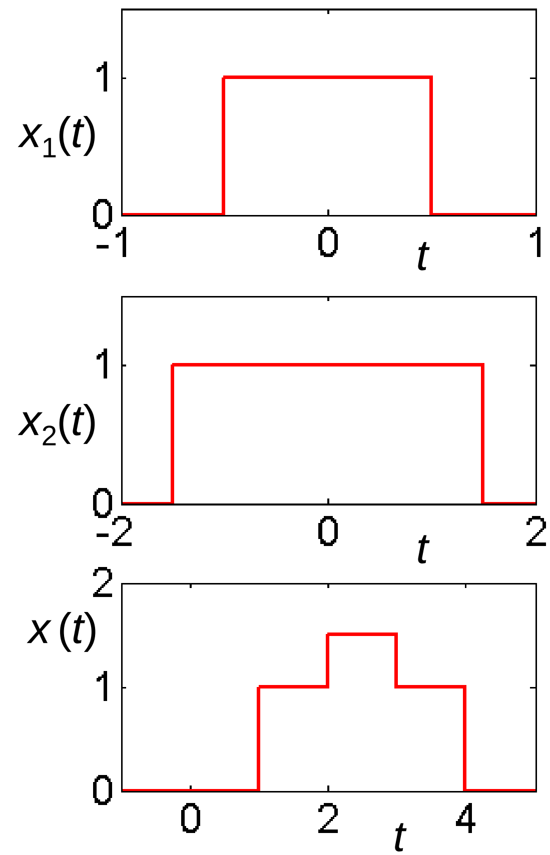
Using the FT of a rectangular pulse L10S7

$$X_1(j\omega) = {2\sin(\omega/2)}\big/{\omega}$$

$$X_2(j\omega) = {2\sin(3\omega/2)}\big/{\omega}$$

Then using the **linearity** and **time shift** Fourier transform properties

$$X(j\omega) = e^{-j5\omega/2}\left(\left(\sin(\omega/2) + 2\sin(3\omega/2)\right)\big/\omega\right)$$



Fourier Transform of a Derivative

By differentiating both sides of the Fourier transform synthesis equation with respect to *t*:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

Therefore noting that this is the synthesis equation for the Fourier transform *jwX*(*jw*)

$$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

This is very important, because it replaces **differentiation** in the **time domain** with **multiplication** (by *jw*) in the **frequency domain**.

We can **solve ODEs** in the **frequency domain** using **algebraic** operations (see next slides)

Convolution in the Frequency Domain

We can easily solve ODEs in the frequency domain:

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

Therefore, to apply **convolution in the frequency domain**, we just have to **multiply** the **two Fourier Transforms**.

To solve for the differential/convolution equation using Fourier transforms:

1. Calculate **Fourier transforms** of $x(t)$ and $h(t)$: $X(jw)$ by $H(jw)$
2. **Multiply** $H(jw)$ by $X(jw)$ to obtain $Y(jw)$
3. Calculate the **inverse Fourier transform** of $Y(jw)$

$H(jw)$ is the LTI system's **transfer function** which is the **Fourier transform** of the **impulse response**, $h(t)$. Very important in the remainder of the course (using Laplace transforms)

This result is proven in the appendix

Example 1: Solving a First Order ODE

Calculate the response of a CT LTI system with impulse response:

$$h(t) = e^{-bt}u(t) \qquad b > 0$$

to the input signal:

$$x(t) = e^{-at}u(t) \qquad a > 0$$

Taking Fourier transforms of both signals:

$$H(j\omega) = \frac{1}{b + j\omega}, \quad X(j\omega) = \frac{1}{a + j\omega}$$

gives the overall frequency response:

$$Y(j\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

to convert this to the time domain, express as **partial fractions**:

$$Y(j\omega) = \frac{1}{b - a}\left(\frac{1}{(a + j\omega)} - \frac{1}{(b + j\omega)}\right)$$

assume
b≠a

Therefore, the CT system response is:

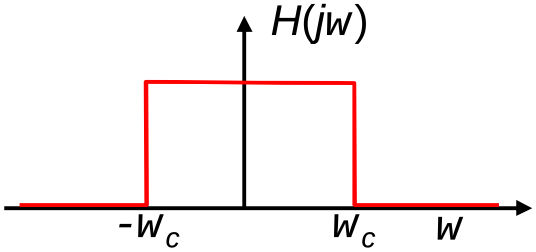
$$y(t) = \tfrac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Example 2: Design a Low Pass Filter

Consider an ideal **low pass filter** in frequency domain:

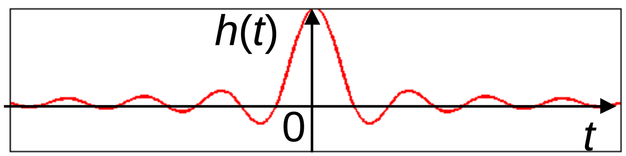
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$Y(j\omega) = \begin{cases} X(j\omega) & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



The **filter's impulse response** is the **inverse Fourier transform**

$$h(t) = \tfrac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{\sin(\omega_c t)}{\pi t}$$



which is an ideal low pass CT filter. However it is non-causal, so this cannot be manufactured exactly & the time-domain oscillations may be undesirable

We need to approximate this filter with a causal system such as 1st order LTI system impulse response {*h*(*t*), *H*(*jw*)}:

$$a^{-1}\frac{\partial y(t)}{\partial t} + y(t) = x(t), \qquad e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

Summary

The Fourier transform is widely used for designing **filters**. You can design systems with reject high frequency noise and just retain the low frequency components. This is natural to describe in the **frequency domain**.

Important **properties** of the Fourier transform are:

1. **Linearity** and **time shifts** $ax(t)+by(t)\overset{F}{\leftrightarrow}aX(j\omega)+bY(j\omega)$
2. **Differentiation** $\frac{dx(t)}{dt}\overset{F}{\leftrightarrow}j\omega X(j\omega)$
3. **Convolution** $y(t)=h(t)*x(t)\overset{F}{\leftrightarrow}Y(j\omega)=H(j\omega)X(j\omega)$

Some operations are **simplified** in the frequency domain, but there are a number of signals for which the Fourier transform does not exist – this leads naturally onto **Laplace transforms**. Similar properties hold for Laplace transforms & the Laplace transform is widely used in engineering analysis.

Appendix: Proof of Convolution Property

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Taking Fourier transforms gives:

$$Y(j\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right) e^{-j\omega t} dt$$

Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right) d\tau$$

By the time shift property, the bracketed term is $e^{-jwt}H(jw)$, so

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(j\omega) d\tau$$

$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= H(j\omega) X(j\omega)$$