Region of Convergence (ROC)

Region of Convergence (ROC) for Laplace Transform

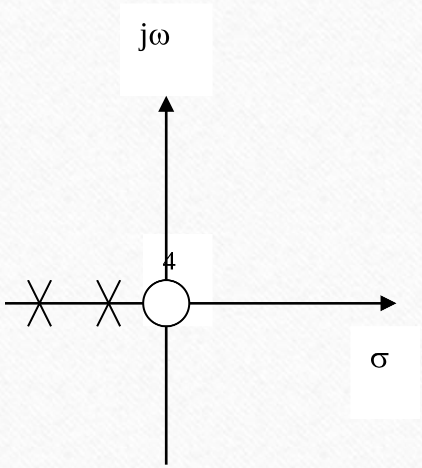
- The *s*-plane refers to the domain where all the possible values of $H(s)$ occur.

- If the polynomial order is 2 with real poles and zeros, the positions of the poles and zeros in the s-plane is shown Figure 1(a).

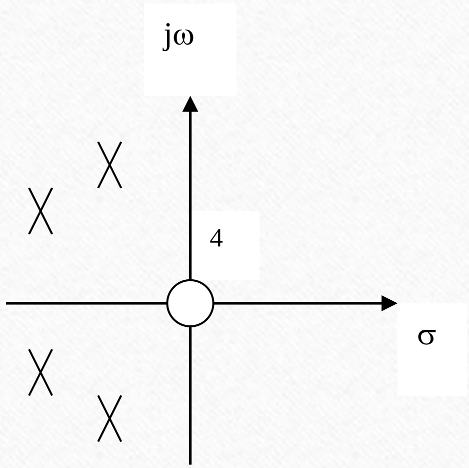
- Similar example for complex poles and zeroes are shown in Figure 1(b).

- For causal systems, the system is stable if the poles are on the right hand side of the s-plane.

ROC



1(a) Real poles and zeros.



2(b) Complex poles and zeros.

Differential Solution and Transfer Function

- Many physical phenomena can be defined as differential equation

$$\frac{d^2}{dt^2}y(t)+a(1)\frac{d}{dt}y(t)+a(2)y(t)=x(t)$$

where $y(t)$ is the output and $x(t)$ is the input. The Laplace transform is one way of solving the relationship between $y(t)$ and $x(t)$.

$$s^2Y(s)+a(1)sY(s)+a(2)Y(s)=X(s)$$

- The transfer function of the system obtained by taking the ratio of $Y(s)$ and $X(s)$ is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + a(1)s + a(2)}$$

- For a more general case, the transfer function is in the form of:

$$H(s) = \frac{b(0)s^N + b(1)s^{N-1} + b(2)s^{N-2} + \dots\dots + b(N)}{s^N + a(1)s^{N-1} + a(2)s^{N-2} + \dots\dots + a(N)} = \frac{\sum_{n=0}^{N} b(n)s^{(N-n)}}{\sum_{n=0}^{N} a(n)s^{(N-n)}}$$

where N is the polynomial order. The transfer function when factorized in terms of its roots is:

$$H(s)=\frac{(s+\beta(0))(s+\beta(1)).......(s+\beta(N))}{(s+\alpha(0))(s+\alpha(1)).......(s+\alpha(N))}=\frac{\prod_{n=0}^{N}(s+\beta(n))}{\prod_{n=0}^{N}(s+\alpha(n))}$$

•The roots in the numerator $-\beta(0)$, $-\beta(1)$ … $-\beta(N)$ are referred as the zeros while the roots in the denominator $-\alpha(0)$, $-\alpha(1)$ … $-\alpha(N)$ are the poles.

Inverse Laplace Transform

- Definition:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds = \frac{1}{2\pi j}\int_{\sigma - jw}^{\sigma + jw} X(s)e^{st}ds$$

- *s* is a complex variable that is defined as $s=\sigma+jw$.
- σ fixed and ω varying from -∞ to ∞.

•we usually use table of Laplace transform pairs to perform Inverse Laplace Transform.

- <u>Example 4</u>

  - Compute x(t) if X(s) = $3/s^2$

$$x(t) = L^{-1}[X(s)] = L^{-1}[\frac{3}{s^2}]$$

$$= 3L^{-1}[\frac{1}{s^2}] = 3t$$

  - Compute f(t) if F(s) = $\frac{2s}{s^2+10}$

f(t)=2 cos $\sqrt{10}t$

- By taking the inverse Laplace transform, the system impulse response $h(t)$ is obtained from $H(s)$.

- If the following transfer function is used as example:

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s^2+a(1)s+a(2)}=\frac{1}{(s+\alpha(0))(s+\alpha(1))}$$

- The application of the partial fraction expansion results in:

$$H(s)=\frac{1}{(s+\alpha(0))(s+\alpha(1))}=\frac{A_0}{(s+\alpha(0))}+\frac{A_1}{(s+\alpha(1))}$$

- From Laplace Transform pairs, the system impulse response, h(t) is:

$$h(t)=A_0e^{-\alpha(0)t}+A_1e^{-\alpha(1)t}$$

- If the poles are complex conjugate pole pairs, then the transfer function is:

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s^2+a(1)s+a(2)}=\frac{1}{(s+\alpha+j\beta)(s+\alpha-j\beta)}$$

- By applying the partial fraction expansion, the transfer function is:

$$H(s)=\frac{1}{(s+\alpha+j\beta)(s+\alpha-j\beta)}=\frac{A_0}{(s+\alpha+j\beta)}+\frac{A_1}{(s+\alpha-j\beta)}$$

- The resulting system impulse response is:

$$h(t)=A_0e^{-\alpha t}e^{j\beta t}+A_1e^{-\alpha t}e^{-j\beta t}$$

•Example 5

A system is defined by a transfer function

$$H(s) = \frac{1}{(s+1)}$$

If the input to the system is

$$x(t) = e^{-0.1t} \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

Output of the system is

Partial fraction expansion

$$Y(s) = H(s)X(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+0.1}\right) = \frac{1}{(s+1)(s+0.1)} = \frac{A_o}{s+1} + \frac{A_1}{s+0.1}$$

$$Y(s) = \frac{-1.1}{s+1} + \frac{1.1}{s+0.1} \longrightarrow y(t) = -1.1e^{-1t} + 1.1e^{-0.1t} \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

•Example 6

A transfer function of a system that has a complex conjugate pole pair is defined as:

$$H(s) = \frac{1}{(s + 0.5 + j2\pi100)(s + 0.5 - j2\pi100)}$$

Partial fraction expansion

Solving the partial fraction expansion results in:

$$H(s) = \frac{1}{(s + 0.5 + j2\pi100)(s + 0.5 - j2\pi100)} = \frac{A_0}{s + 0.5 + j2\pi100} + \frac{A_1}{s + 0.5 - j2\pi100}$$

$$h(t) = \left(-\frac{1}{j4\pi100}\right)e^{-0.5t}e^{-j2\pi100t} + \left(\frac{1}{j4\pi100}\right)e^{-0.5t}e^{j2\pi100t}$$

$$= \left(\frac{1}{2\pi100}\right)e^{-0.5t}\sin(2\pi100t) \qquad t \geq 0$$

$$= 0 \qquad t < 0$$

RELATIONSHIP BETWEEN LAPLACE AND FOURIER TRANSFORM

- The Fourier transform can be obtained from the Laplace transform by making the substitution $s = jw = j2\pi f$.

- A transfer function is

$$H(s) = \frac{1}{\left(s^2 + s + (2\pi 100)^2\right)}$$

By making the substitution $s = jw = j2\pi f$, the Fourier transform of the transfer function is

$$H(f) = \frac{1}{\left(-(2\pi f)^2 + j2\pi f + (2\pi 100)^2\right)} = \frac{1}{\left(\left((2\pi 100)^2 - (2\pi f)^2\right) + j2\pi f\right)}$$

RELATIONSHIP BETWEEN LAPLACE AND FOURIER TRANSFORM

- The Fourier transform when defined in terms of the magnitude and phase is

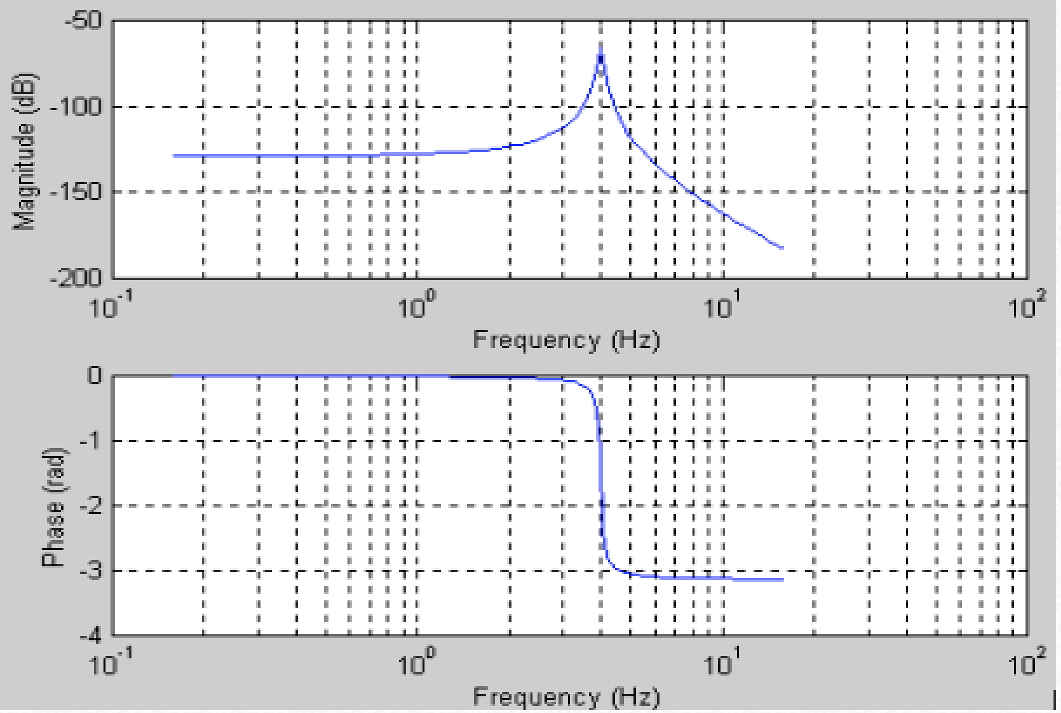
$$|H(f)| = \frac{1}{\sqrt{\left((2\pi 100)^2 - (2\pi f)^2\right)^2 + (2\pi f)^2}}$$

Magnitude

$$\phi(f) = -\tan^{-1}\left(\frac{2\pi f}{(2\pi 100)^2 - (2\pi f)^2}\right)$$

Phase

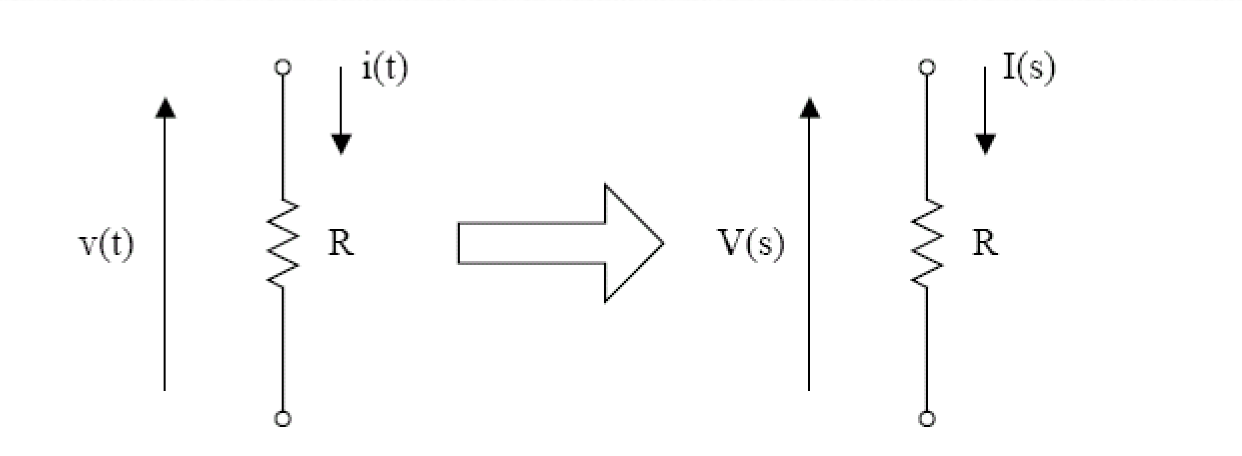
FREQUENCY RESPONSE OF TRANSFER FUNCTION



Magnitude and phase plot of transfer function

Circuit Analysis using Laplace Transform

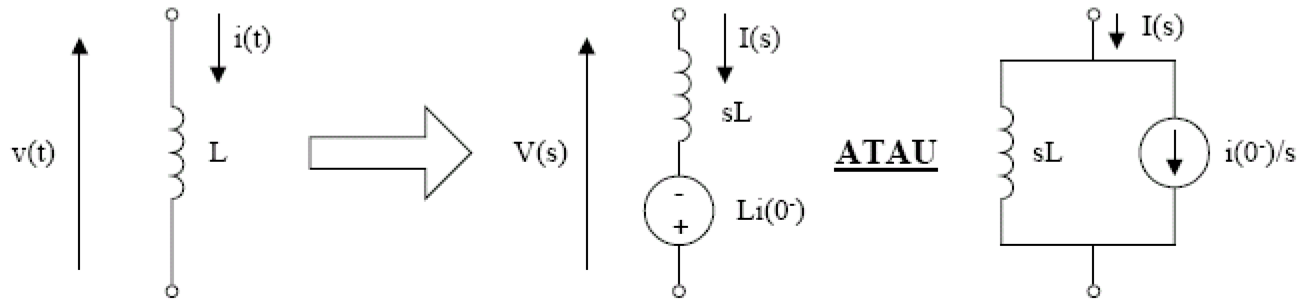
a) Resistor



Time domain S-domain

$$v(t) = i(t)\,R \quad \Leftrightarrow \quad V(s) = I(s)\,R$$

b) Inductor

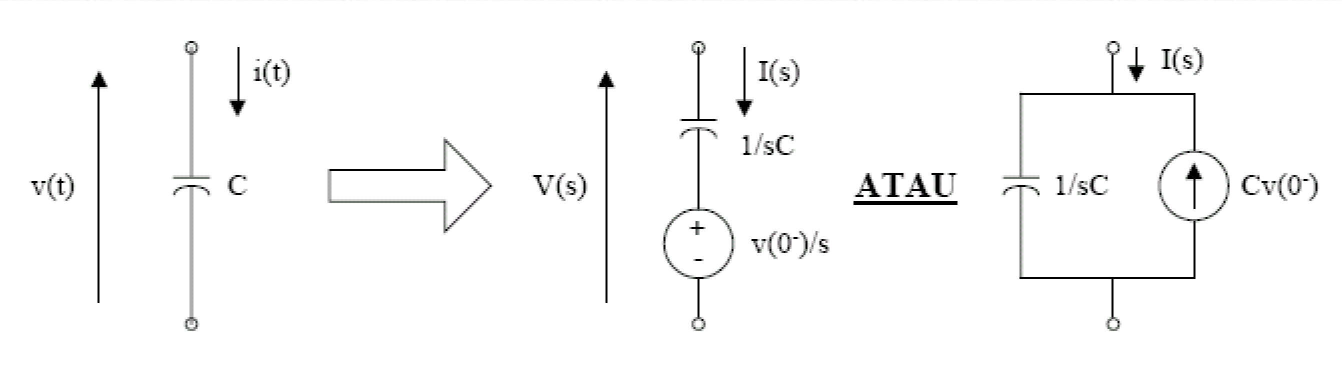


Time domain S-domain

$$v(t) = L\frac{di(t)}{dt} \quad \Leftrightarrow \quad V(s) = s\,L\,I(s) - L\,i(0^-)$$

$$I(s) = \frac{1}{sL}V(s) + \frac{1}{s}i(0^-)$$

b) Capacitor



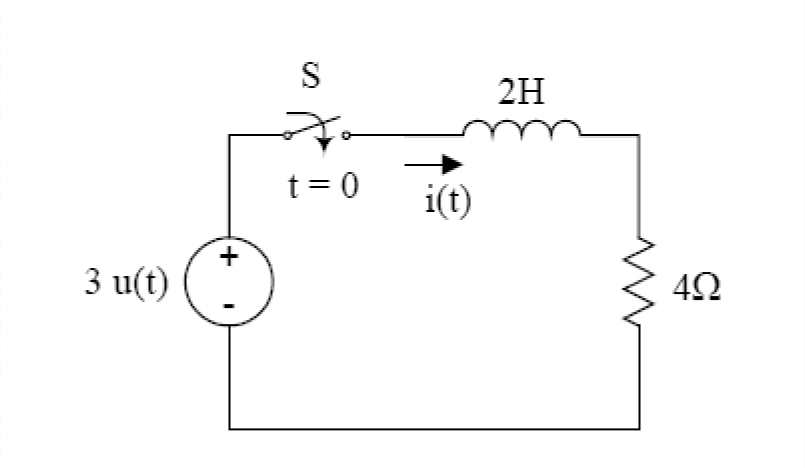
Time domain S-domain

$$v(t) = \frac{1}{C}\int i(t)\,dt \quad \Leftrightarrow \quad V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

$$i(t) = C\frac{dv(t)}{dt} \qquad I(s) = sC\,V(s) - C\,v(0^-)$$

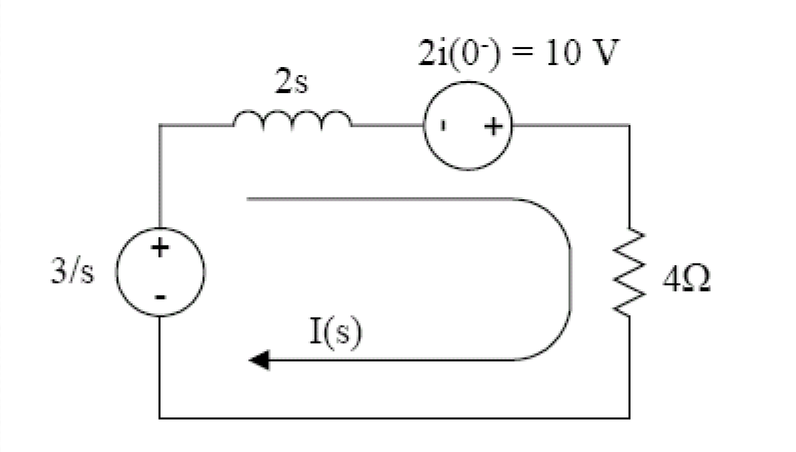
<u>Example 7</u>

Determine i(t) when switch S in the circuit was closed when t≥0s. Given the the start value of i(t), $i(0^-)$=5A.



Solution

The s-domain circuit is:



$$(2s + 4)\, I(s) = \frac{3}{s} + 10$$

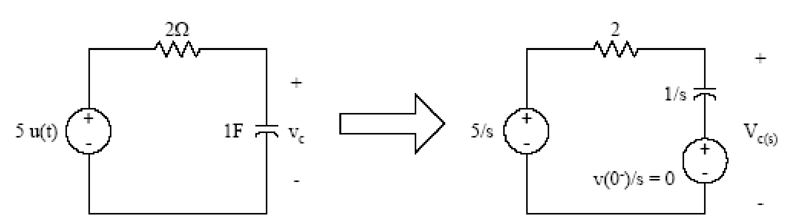
$$I(s) = \frac{3 + 10s}{s(4 + 2s)}$$

Inverse laplace transform

$$\therefore \quad i(t) = 0.75\, u(t) + 4.25\, e^{-2t}\, u(t)$$

Solution

- At 0≤t≤1



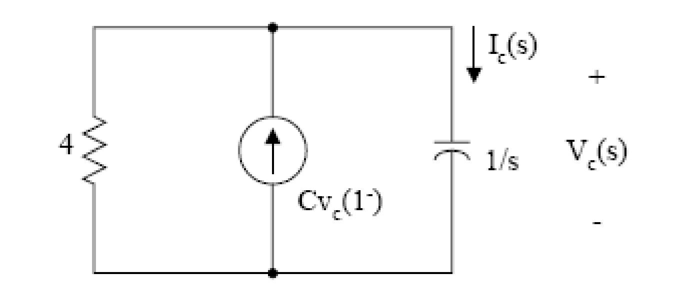
$$Vc(S) = \frac{1/S}{2+1/S}\left(\frac{5}{S}\right)$$
$$= \frac{5}{(2S+1)S}$$
$$= \frac{2.5}{S(S+0.5)}$$
$$= \frac{A}{S} + \frac{B}{S+0.5}$$

$$A = \left.\frac{2.5}{S+0.5}\right|_{S=0} = 5$$
$$B = \left.\frac{2.5}{S}\right|_{S=0.5} = -5$$

$$Vc(t) = 5u(t) - 5e^{-0.5t}u(t)$$

- At t≥1

$$Vc(1-) = 5u(t) - 5e^{-0.5t}u(t)$$
$$= 5 - 5e^{-0.5}$$
$$= 1.967V$$



$$Ic(S) = \frac{4}{4+1/S}(1.967)$$
$$= \frac{S(1.967)}{S+0.25}$$

$$Vc(S) = \frac{Ic(S)}{S}$$
$$= \frac{1.967}{S+0.25}$$

- For t'≥1 where t'=t-1

$$Vc(t') = 1.967e^{-0.25t'}u(t)$$
$$Vc(t-1) = 1.967e^{-0.25(t-1)}u(t-1)$$