Laplace Transform Properties

Table for Some Laplace Transform Pairs:

Signal, x(t)	Transform, X(s)
u(t)	1/s
δ(t)	1
t	$1/s^2$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
sin2πft	$\frac{2\pi f}{s^2+(2\pi f)^2}$
cos2πft	$\frac{s}{s^2+(2\pi f)^2}$

Laplace Transform Properties:

Properties	x(t)	X(s)
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Shift in time	$x(t-\tau)$	$e^{-s\tau}X(s)$
Shift in s-plane	$e^{at}x(t)$	$X(s-a)$
Time scaling	$x(at)$	$\frac{1}{a}X(\frac{s}{a})$
Differentiation	$\frac{dx}{dt}$	$sX(s)-x(0^-)$
Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(s)}{s}+\int_{-\infty}^{0}x(\tau)d\tau$
Multiplication by t	$tx(t)$	$-\frac{dX(s)}{ds}$
Division by t	$\frac{x(t)}{t}$	$\int_{s}^{\infty}X(u)du$
Convolution in time	$\int_{-\infty}^{\infty}x_1(\tau)x_2(t-\tau)d\tau$	$X_1(s)X_2(s)$
Convolution in frequency	x_1x_2	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}X_1(u)X_2(s-u)du$
Initial value theorem	$x(0^+)$	$\lim_{s\to\infty}sX(s)$
Final value theorem	$\lim_{t\to\infty}x(t)$	$\lim_{s\to 0}sX(s)$ (Valid only if sX(s) has all its poles in the left half of the s-plane.)

Laplace Transform Properties

Linearity :

$$\mathcal{L}\,[\,a\,f_1(t) + b\,f_2(t)\,] = a\,F_1(s) + b\,F_2(s)$$

Pembuktian:

$$\mathcal{L}\,[\,af_1(t) + bf_2(t)\,] = \int_0^\infty [\,af_1(t) + bf_2(t)\,]\,e^{-st}\,dt$$

$$= a\int_0^\infty f_1(t)\,e^{-st}\,dt + b\int_0^\infty f_2(t)\,e^{-st}\,dt$$

$$= a\,F_1(s) + b\,F_2(s)$$

Example of Linearity :

$$f(t) = \cosh at = \frac{e^{at} + e^{-at}}{2}.$$

$$f(t) = \frac{1}{2} e^{at} + \frac{1}{2} e^{-at}$$

$$\therefore \quad F(s) = \frac{1}{2} L[e^{at}] + \frac{1}{2} L[e^{-at}]$$

$$= \frac{1}{2(s - a)} + \frac{1}{2(s + a)} = \frac{s}{s^2 - a^2}$$

Differentiation :

- If f(t), t≥0 the differentiation of f(t) is df(t)/dt, the Laplace transform is:

$$L\left[\frac{df}{dt}\right] = s\,F(s) - f(0)$$

Where f(0) is f(t) at t=0. Start value.

Pembuktian:

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= s\,F(s) - f(0)$$

•For higher level differentiation:

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s\,\mathcal{L}\left[\frac{df(t)}{dt}\right] - \frac{df(0)}{dt}$$

$$= s\left[s\,F(s) - f(0)\right] - \frac{df(0)}{dt}$$

$$= s^2 F(s) - s\,f(0) - \frac{df(0)}{dt}$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\frac{df(0)}{dt} - \ldots..$$

$$\ldots. - s\frac{d^{n-2} f(0)}{dt^{n-2}} - \frac{d^{n-1} f(0)}{dt^{n-1}}$$

Integration :

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\int_0^t f(t)\,dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \frac{F(s)}{s}$$

Time shifting :

If Laplace transform of f(t) is F(s), then:

$$\mathcal{L}\left[f(t - a) \right] = e^{-as} F(s)$$

Frequency shifting :

If Laplace transform of f(t) is F(s), then:

$$\mathcal{L}\left[e^{at} f(t) \right] = F(s - a)$$

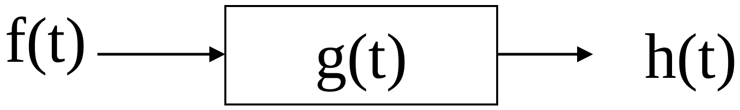
Exponential Multiplication :

$$\mathcal{L}\left[e^{at} x(t) \right] = X(s - a)$$

Pembuktian →

$$\int_0^\infty e^{at} x(t) e^{-st} dt = \int_0^\infty x(t) e^{-(s-a)t} dt = X(s - a)$$

Convolution :



Convolution in time domain :

$$h(t) = f(t) * g(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau$$

Laplace Transform :

$$H(s)=F(s)G(s)$$