Properties of Z transform, Inverse Z Transform & Solution of Difference Equations

Properties of the z-Transform

- **Linearity:** $ax_1[n]+bx_2[n] \Leftrightarrow aX_1[z]+bX_2[z]$
- **Time-shift:** $x[n-n_0] \Leftrightarrow z^{-n_0}X[z]$
- **Multiplication by n:** $nx[n] \Leftrightarrow -z\dfrac{dX[z]}{dz}$

Proof: $X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$\frac{dX(z)}{dz} = -n\sum_{n=-\infty}^{\infty} x[n]z^{-n-1} \Rightarrow -z\frac{dX(z)}{dz} = n\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathbf{Z}\{nx[n]\}$$

- **Multiplication by a^n:** $a^n x[n] \Leftrightarrow X\left(\dfrac{z}{a}\right)$

Proof: $\mathbf{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty}(a^n x[n])z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\dfrac{z}{a}\right)^{-n} = X\left(\dfrac{z}{a}\right)$

- **Multiplication by $e^{j\omega n}$:** $e^{j\omega n}x[n] \Leftrightarrow X\left(e^{-j\omega n}z\right)$
- **Multiplication by $\cos\omega n$:** $\cos(\omega n)x[n] \Leftrightarrow (1/2)\left[X\left(e^{j\omega n}z\right)+X\left(e^{-j\omega n}z\right)\right]$
- **Multiplication by $\sin\omega n$:** $\sin(\omega n)x[n] \Leftrightarrow (j/2)\left[X\left(e^{j\omega n}z\right)-X\left(e^{-j\omega n}z\right)\right]$
- **Summation:** $v[n]=\sum_{i=0}^{n} x[i] \Leftrightarrow V(z)=\dfrac{1}{1-z^{-1}}X(z)$

Convolution

- **Convolution:** $x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \quad \Leftrightarrow \quad X(z)H(z)$

Proof: $$\mathbf{Z}\left[x[n]*h[n]\right]=\mathbf{Z}\left[\sum_{k=-\infty}^{\infty}x[k]h[n-k]\right]=\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}x[k]h[n-k]\right]z^{-n}$$

$$=\sum_{k=-\infty}^{\infty}x[k]\left[\sum_{n=-\infty}^{\infty}h[n-k]z^{-n}\right]$$

Change of index on the second sum: $m=n-k$

$$\mathbf{Z}\left[x[n]*h[n]\right]=\sum_{k=-\infty}^{\infty}x[k]\left[\sum_{m=-\infty}^{\infty}h[m]z^{-(m+i)}\right]=\left[\sum_{k=-\infty}^{\infty}x[k]z^{-k)}\right]\left[\sum_{m=-\infty}^{\infty}h[m]z^{-m}\right]$$

$$=X(z)H(z)$$

The ROC is at least the intersection of the ROCs of $x[n]$ and $h[n]$, but can be a larger region if there is pole/zero cancellation.

- **The system transfer function is completely analogous to the CT case:**

$$h[n] \quad \Leftrightarrow \quad H(z)=\sum_{n=-\infty}^{\infty}h[n]z^{-n}$$

- **Causality:** $h[n]=0 \quad n\le 0$

Implies the ROC must be the exterior of a circle and include $z = \infty$.

Initial-Value and Final-Value Theorems (One-Sided ZT)

- **Initial Value Theorem:** $x[0] = \lim_{z\to\infty} X(z)$

  Proof: $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = \lim_{z\to\infty} x[0] + x[1]z^{-1} + \ldots = x[0]$

- **Final Value Theorem:** $\lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$

- **Example:** $X(z) = \dfrac{3z^2 - 2z + 4}{z^3 - 2z^2 + 1.5z - 0.5} = \dfrac{3z^2 - 2z + 4}{(z-1)(z^2 - z + 0.5)}$

  $$\lim_{n\to\infty} x[n] = [(z-1)X(z)]\Big|_{z=1} = \left.\frac{3z^2 - 2z + 4}{z^2 - z + 0.5}\right|_{z=1} = \frac{5}{.5} = 10$$

- **Tables 7.2 and 7.3 in the textbook contain a summary of the *z*-Transform properties and common transform pairs.**

Inverse Laplace Transform

- **Recall the definition of the inverse Laplace transform via contour integration:**

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds = \frac{1}{2\pi j}\oint_C X(s)e^{st}ds$$

- **The inverse z-transform follows from this:**

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Evaluation of this integral is beyond the scope of this course. Instead, as with the Laplace transform, we will restrict our interest in the inverse transform to rational forms (ratio of polynomials). We will see shortly that this is convenient since linear constant-coefficient difference equations can be converted to polynomials using the z-transform.

- **As with the Laplace transform, there are two common approaches:**
  - **Long Division**
  - **Partial Fractions Expansion**
- **Expansion by long division Is also known as the power series expansion approach and can be easily demonstrated by an example.**

Long Division

- **Consider:** $X(z) = \dfrac{z^2 - 1}{z^3 + 2z + 4}$

Solution:

$$\begin{array}{r|l} & z^{-1} \\ \hline z^3+2z+4 & z^2 - 1 \\ & \underline{z^2 + 2 + 4z^{-1}} \\ & -3 - 4z^{-1} \end{array}$$

$$\begin{array}{r|l} & z^{-1} + 0z^{-2} - 3z^{-3} \\ \hline z^3+2z+4 & z^2 - 1 \\ & \underline{z^2 + 2 + 4z^{-1}} \\ & -3 - 4z^{-1} \\ & \underline{-3 \qquad -6z^{-2} - 12z^{-3}} \\ & -4z^{-1} + 6z^{-2} + 12z^{-3} \end{array}$$

$$\begin{array}{r|l} & z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} \\ \hline z^3+2z+4 & z^2 - 1 \\ & \underline{z^2 + 2 + 4z^{-1}} \\ & -3 - 4z^{-1} \\ & \underline{-3 \qquad -6z^{-2} - 12z^{-3}} \\ & -4z^{-1} + 6z^{-2} + 12z^{-3} \\ & \underline{-4z^{-1} \qquad - 8z^{-3} - 16z^{-4}} \\ & 6z^{-2} + 20z^{-3} + 16z^{-4} \end{array}$$

$$\therefore \quad X(z) = z^{-1} + 0z^{-2} - 3z^{-3} - 4z^{-4} + \ldots$$

$$\Rightarrow x[n] = 0\delta[n] + 1\delta[n-1] - 3\delta[n-3] - 4\delta[n-4] + \ldots$$

Implications of stability?

Inverse z-Transform Using Partial Fractions

- **Rational transforms can be factored using the same partial fractions approach we used for the Laplace transforms.**
- **The partial fractions approach is preferred if we want a closed-form solution rather than the numerical solution long division provides.**
- **Example:** $X(z) = \dfrac{z^3+1}{z^3 - z^2 - z - 2}$

  In this example, the order of the numerator and denominator are the same. For this case, we can use a trick of factoring $X(z)/z$:

$$A(z) = z^3 - z^2 - z - 2 = (z-2)(z+0.5+j0.866)(z+0.5-j0.866)$$

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z+0.5+j0.866} + \frac{\bar{c}_1}{z+0.5-j0.866} + \frac{c_3}{z-2}$$

$$c_0 = \left[\frac{X(z)}{z}(z)\right]_{z=0} = \frac{1}{-2} = -0.5$$

$$c_1 = \left[\frac{X(z)}{z}\left(z+0.5+j0.866\right)\right]_{z=-0.5-j0.866} = 0.429 + j0.0825$$

$$c_3 = \left[\frac{X(z)}{z}\left(z-2\right)\right]_{z=2} = 0.643$$

Inverse z-Transform (Cont.)

We can compute the inverse using our table of common transforms:

$$X(z) = c_0 + \frac{c_1 z}{z + 0.5 + j0.866} + \frac{\bar{c}_1 z}{z + 0.5 - j0.866} + \frac{c_3 z}{z - 2}$$

$$= c_0 + \frac{c_1}{1 + 0.5 + j0.866 z^{-1}} + \frac{\bar{c}_1}{1 + 0.5 - j0.866 z^{-1}} + \frac{c_3}{1 - 2z^{-1}}$$

$$x[n] = c_0 \delta[n] + c_1(-0.5 - j0.866)^n u[n] + \bar{c}_1(-0.5 + j0.866)^n u[n] + c_3 2^n u[n]$$

The exponential terms can be converted to a single cosine using a magnitude/phase conversion:

$$|p_1| = \sqrt{(0.5)^2 + (0.866)^2} = 1$$

$$\angle p_1 = \pi + \tan^{-1}\frac{0.866}{0.5} = \frac{4\pi}{3} \text{ rad}$$

$$|c_1| = \sqrt{(0.429)^2 + (0.0825)^2} = 0.437$$

$$\angle c_1 = \tan^{-1}\frac{0.0825}{0.429} = 0.19 \text{ rad} \quad (10.89°)$$

$$x[n] = c_0 \delta[n] + c_1(-0.5 - j0.866)^n u[n] + \bar{c}_1(-0.5 + j0.866)^n u[n] + c_3 2^n u[n]$$

$$= c_0 \delta[n] + 2|c_1||p_1| \cos(\angle p_1 n + \angle c_1) + c_3 (2)^n u[n]$$

$$= -0.5\delta[n] + 0.874 \cos(\frac{4\pi}{3} n + 0.19) + 0.643(2)^n u[n]$$

First-Order Difference Equations

- Consider a first-order difference equation:

$$y[n] + ay[n-1] = bx[n]$$

- We can apply the time-shift property:

$$Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)$$

- We can solve for $Y(z)$:

$$Y(z) = -\frac{ay[-1]}{1+az^{-1}} + \frac{b}{1+az^{-1}}X(z)$$

- The response is again a function of two things: the response due to the initial condition and the response due to the input.

- If the initial condition is zero:

$$Y(z) = \frac{b}{1+az^{-1}}X(z) \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1+az^{-1}}$$

- Applying the inverse z-Transform:

$$h[n] = \mathbf{Z}^{-1}\left[\frac{b}{1+az^{-1}}\right] = b(-a)^n u[n]$$

- Is this system causal? Why?

- Is this system stable? Why?

- Suppose the input was a sinusoid. How would you compute the output?

Example of a First-Order System

- **Consider the unit-step response of this system:**

$$x[n] = u[n] \quad \Rightarrow \quad X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - a}$$

$$Y(z) = -\frac{ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}} X(z) = -\frac{ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}} \left(\frac{1}{1 - z^{-1}} \right)$$

$$= -\frac{ay[-1]z}{z + a} + \frac{bz^2}{(z + a)(z - 1)}$$

- **Use the (1/z) approach for the inverse transform:**

$$\frac{V(z)}{z} = \left(\frac{1}{z} \right) \frac{bz^2}{(z + a)(z - 1)} = \frac{ab/(1 + a)}{z + a} + \frac{b/(1 + a)}{z - 1}$$

$$Y(z) = -\frac{ay[-1]z}{z + a} + \frac{ab/(1 + a)z}{z + a} + \frac{b/(1 + a)z}{z - 1} = -\frac{ay[-1]z}{z + a} + \frac{b}{1 + a} \left(\frac{az}{z + a} + \frac{z}{z - 1} \right)$$

$$y[n] = -ay[-1](-a)^n + \frac{b}{1 + a} [a(-a)^n + (1)^n]$$

$$= -ay[-1](-a)^n + \frac{b}{1 + a} [-(-a)^{n+1} + 1], \quad n = 0, 1, 2, \ldots$$

- **The output consists of a DC term, an exponential term due to the I.C., and an exponential term due to the input. Under what conditions is the output stable?**

Second-Order Difference Equations

- **Consider a second-order difference equation:**

$$y[n]+a_1 y[n-1]+a_2 y[n-2]=b_0 x[n]+b_1 x[n-1]$$

- **We can apply the time-shift property:**

$$Y(z)+a_1\left[z^{-1}Y(z)+y[-1]\right]+a_2\left[z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\right]=b_0 X(z)+b_1 z^{-1}X(z)$$

- **Assume** *x*[-*1*] = *0* **and solve for** *Y*(*z*)**:**

$$Y(z)=\frac{a_2 y[-2]-a_1 y[-1]-a_2 y[-1]z^{-1}}{1+a_1 z^{-1}+a_2 z^{-2}}+\frac{b_0+b_1 z^{-1}}{1+a_1 z^{-1}+a_2 z^{-2}}X(z)$$

- **Multiplying** z^2/z^2**:**

$$Y(z)=\frac{-\left(a_1 y[-1]+a_2 y[-2]\right)z^2-a_2 y[-1]z}{z^2+a_1 z+a_2}+\frac{b_0 z^2+b_1 z}{z^2+a_1 z+a_2}X(z)$$

- **Assuming the initial conditions are zero:**

$$Y(z)=\frac{b_0 z^2+b_1 z}{z^2+a_1 z+a_2}$$

- **Note that the impulse response is of the form:**

$$h[n]=\left(a^n\cos\omega n\right)u[n]\quad\Leftrightarrow\quad H(z)=\frac{z^2-(a\cos\omega)z}{z^2-(2a\cos\omega)z+a^2}$$

This can be visualized as a complex pole pair with a center frequency and bandwidth (see Java applet).

Example of a Second-Order System

- **Consider the unit-step response of this system:**

$$x[n] = u[n] \quad \Rightarrow \quad X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - a}$$

$$y[n] + 1.5y[n-1] + 0.5y[n-2] = x[n] - x[n-1] \quad \text{where} \quad y[-1] = 2,\ y[-2] = 1$$

$$Y(z) = \frac{-\left(a_1 y[-1] + a_2 y[-2]\right)z^2 - a_2 y[-1]z}{z^2 + a_1 z + a_2} + \frac{b_0 z^2 + b_1 z}{z^2 + a_1 z + a_2} X(z)$$

$$= \frac{-\left((1.5)(2) + (0.5)(1)\right)z^2 - (0.5)(2)z}{z^2 + 1.5z + 0.5} + \frac{z^2 - z}{z^2 + 1.5z + 0.5}\left(\frac{z}{z-1}\right)$$

$$= \frac{-3.5z^2 - z}{z^2 + 1.5z + 0.5} + \frac{z^2}{z^2 + 1.5z + 0.5} \qquad [\text{note}: (z^2 - z)z = z^2(z-1)]$$

- **We can further simplify this:**

$$Y(z) = \frac{-2.5z^2 - z}{z^2 + 1.5z + 0.5}$$

$$= \frac{0.5z}{z + 0.5} - \frac{3z}{z + 1}$$

- **The inverse *z*-transform gives:**

$$y[n] = 0.5(-0.5)^n - 3(-1)^n, \quad n = 0, 1, 2, \ldots$$

Nth-Order Difference Equations

- **Consider a general difference equation:**

$$y[n]+\sum_{i=1}^{N} a_i y[n-i]=\sum_{i=1}^{M} b_i x[n-i]$$

- **We can apply the time-shift property once again:**

$$Y(z)+\sum_{i=1}^{N} a_i z^{-i} Y(z)=\sum_{i=1}^{M} b_i z^{-i} X(z) \qquad \text{(assuming zero initial conditions)}$$

$$Y(z)\left[1+\sum_{i=1}^{N} a_i z^{-i}\right]=X(z)\left[\sum_{i=1}^{M} b_i z^{-i}\right]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{i=1}^{M} b_i z^{-i}}{1+\sum_{i=1}^{N} a_i z^{-i}}=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}+\ldots+b_M z^{-M}}{1+a_0+a_1 z^{-1}+a_2 z^{-2}+\ldots+a_N z^{-N}}$$

- **We can again see the important of poles in the stability and overall frequency response of the system.**
- **Since the coefficients of the denominator are most often real, the transfer function can be factored into a product of complex conjugate poles, which in turn means the impulse response can be computed as the sum of damped sinusoids. Why?**
- **The frequency response of the system can be found by setting $z = e^{j\omega}$.**

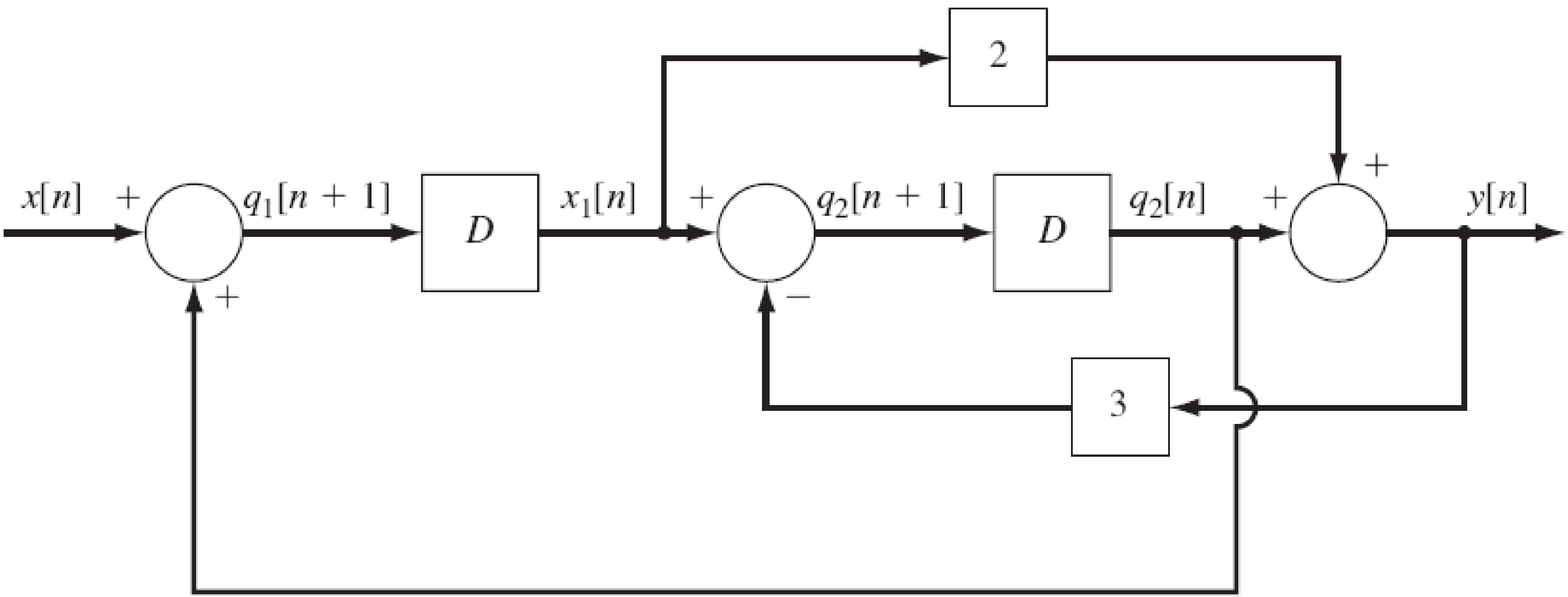
Transfer Functions

- **In addition to our normal transfer function components, such as summation and multiplication, we use one important additional component: delay.**
- **This is often denoted by its z-transform equivalent.**
- **We can illustrate this with an example (assume initial conditions are zero):**

$x[n]$ **D** $y[n] = x[n-1]$

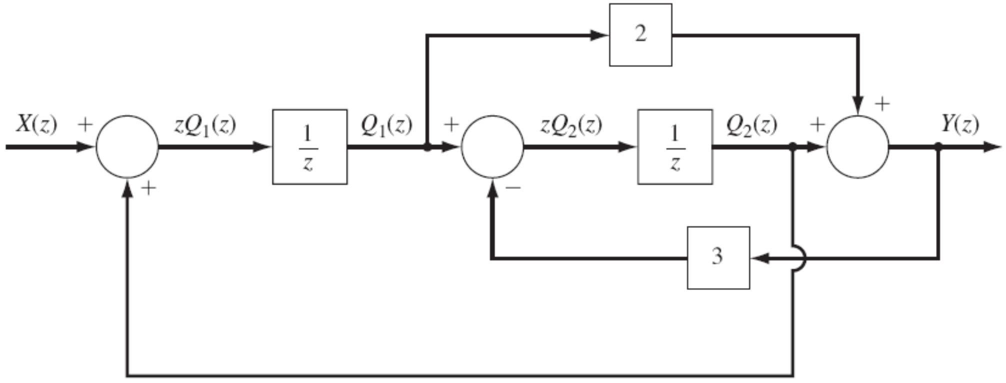
$x[n]$ $\mathbf{z^{-1}}$ $y[n] = x[n-1]$

$$Y(z) = z^{-1}X(z)$$



Transfer Function Example

- **Redraw using *z*-transform:**



- **Write equations for the behavior at each of the summation nodes:**

$$zQ_1(z) = Q_2(z) + X(z)$$
$$zQ_2(z) = Q_1(z) - 3Y(z)$$
$$Y(z) = 2Q_1(z) + Q_2(z)$$

- **Three equations and three unknowns: solve the first for $Q_1(z)$ and substitute into the other two equations.**

$$Q_1(z) = z^{-1}Q_2(z) + z^{-1}X(z)$$
$$zQ_2(z) = \left[z^{-1}Q_2(z) + z^{-1}X(z)\right] - 3Y(z)$$
$$Q_2(z) = z^{-2}Q_2(z) + z^{-2}X(z) - 3z^{-1}Y(z)$$
$$Q_2(z) = \frac{1}{1-z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]$$
$$Y(z) = 2z^{-1}\left[\frac{1}{1-z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]\right] + 2z^{-1}X(z) + \frac{1}{1-z^{-2}}\left[z^{-2}X(z) - 3z^{-1}Y(z)\right]$$

Simplify. . .

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z+1}{z^2+3z+5}$$

Basic Interconnections of Transfer Functions