SYSTEM POLES AND ZEROS

1 System Poles and Zeros

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable $s = \sigma + j\omega$, that is

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \tag{1}$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2)\ldots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_{n-1})(s - p_n)}, \tag{2}$$

where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients defined by the system's differential equation and $K = b_m/a_n$. As written in Eq. (2) the z_i's are the roots of the equation

$$N(s) = 0, \tag{3}$$

and are defined to be the system *zeros*, and the p_i's are the roots of the equation

$$D(s) = 0, \tag{4}$$

and are defined to be the system *poles*. In Eq. (2) the factors in the numerator and denominator are written so that when $s = z_i$ the numerator $N(s) = 0$ and the transfer function vanishes, that is

$$\lim_{s \to z_i} H(s) = 0.$$

and similarly when $s = p_i$ the denominator polynomial $D(s) = 0$ and the value of the transfer function becomes unbounded,

$$\lim_{s \to p_i} H(s) = \infty.$$

All of the coefficients of polynomials $N(s)$ and $D(s)$ are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs. In general for the poles, either $p_i = \sigma_i$, or else $p_i, p_{i+1} = \sigma_i \pm j\omega_i$. The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial $D(s)$. Similarly, the system zeros are either real or appear in complex conjugate pairs.

■ Example

A linear system is described by the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + 1.$$

Find the system poles and zeros.

Solution: From the differential equation the transfer function is

$$H(s) = \frac{2s+1}{s^2+5s+6}. \tag{5}$$

which may be written in factored form

$$\begin{aligned} H(s) &= \frac{1}{2}\frac{s+1/2}{(s+3)(s+2)} \\ &= \frac{1}{2}\frac{s-(-1/2)}{(s-(-3))(s-(-2))}. \end{aligned} \tag{6}$$

The system therefore has a single real zero at $s = -1/2$, and a pair of real poles at $s = -3$ and $s = -2$.

The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics. Together with the gain constant K they completely characterize the differential equation, and provide a complete description of the system.

■ Example

A system has a pair of complex conjugate poles $p_1, p_2 = -1 \pm j2$, a single real zero $z_1 = -4$, and a gain factor $K = 3$. Find the differential equation representing the system.

Solution: The transfer function is

$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} \\ &= 3\frac{(s+4)}{s^2+2s+5} \end{aligned} \tag{7}$$

and the differential equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 3\frac{du}{dt} + 12u \tag{8}$$

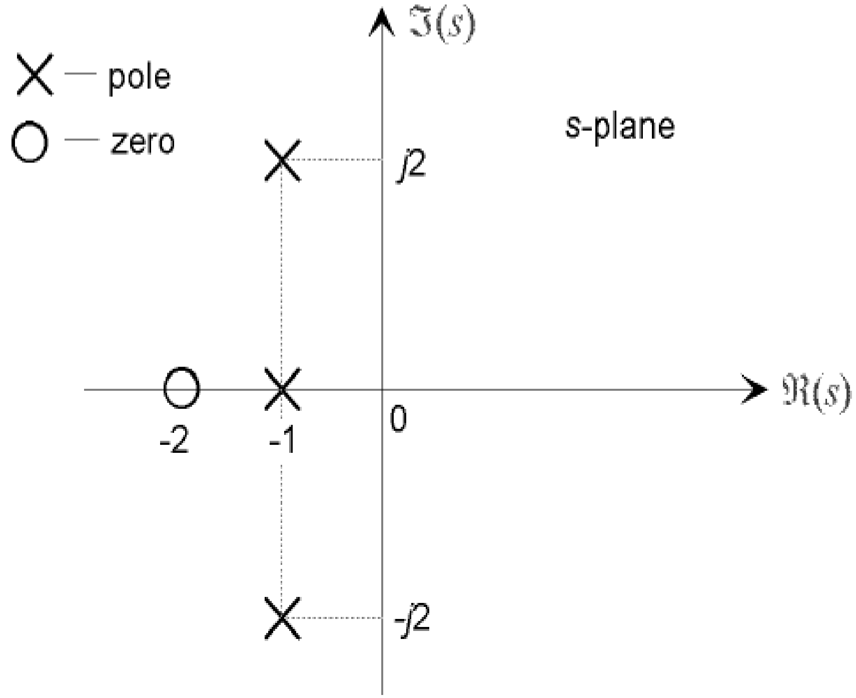

Figure 1: The pole-zero plot for a typical third-order system with one real pole and a complex conjugate pole pair, and a single real zero.

Regions in Pole-Zero Plot

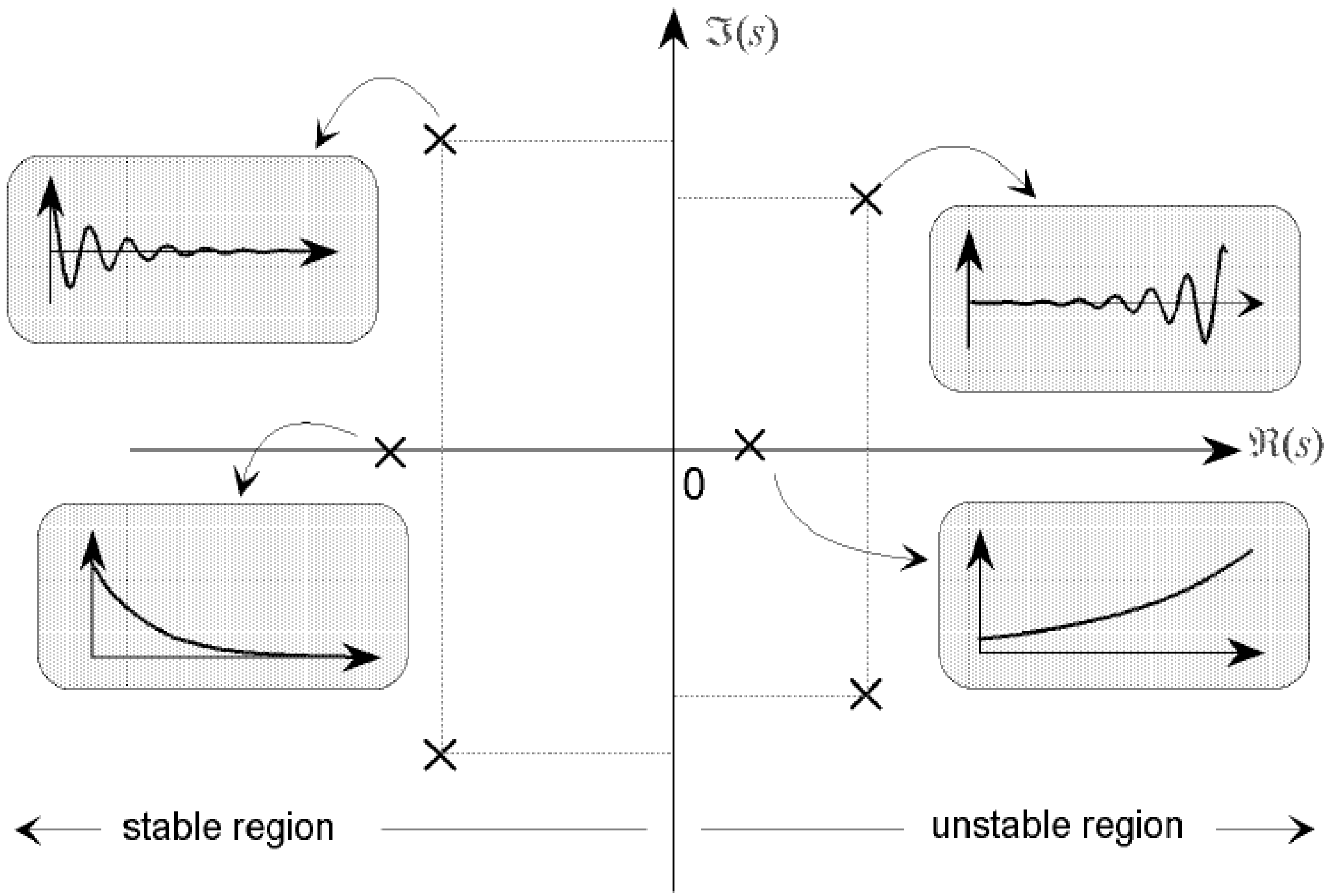

Figure 2: The specification of the form of components of the homogeneous response from the system pole locations on the pole-zero plot.