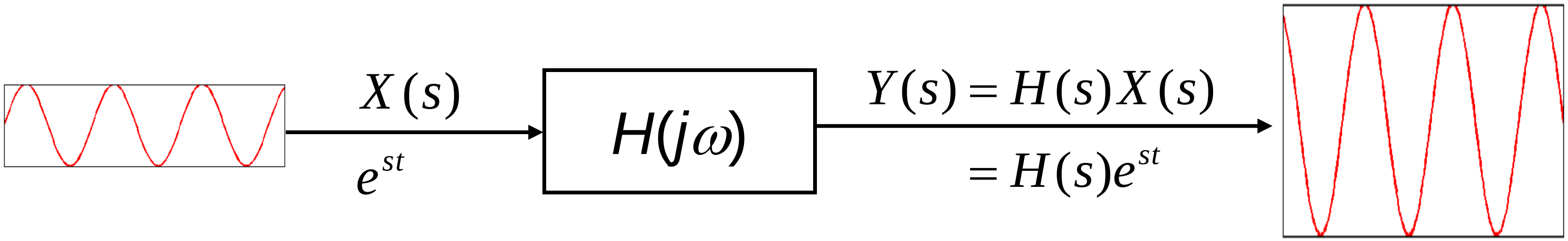
Continuous-Time SystemTransfer Functions

Review: Transfer Functions, Frequency Response & Poles and Zeros



The system's transfer function is the Laplace (Fourier) transform of the system's impulse response $H(s)$ ($H(j\omega)$).

The transfer function's poles and zeros are $H(s) \propto \Pi_i(s-z_i)/\Pi_j(s-p_i)$.

This enables us to both calculate (from the differential equations) and analyse a system's response

Frequency response magnitude/phase decomposition

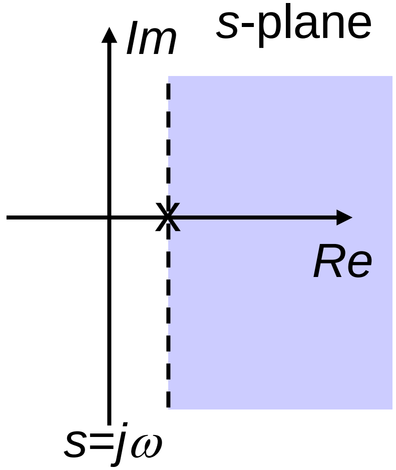
$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

Bode diagrams are a log/log plot of this information

System Causality & Transfer Functions

Remember, a system is causal if $y(t)$ only depends on $x(t)$, $dx(t)/dt$,…,$x(t-T)$ where $T>0$

This is equivalent to saying that an LTI system's impulse is $h(t) = 0$ whenever $t<0$.



Theorem The ROC associated with the (Laplace) transfer function of a causal system is a right-half plane

Note the converse is not necessarily true (but **is true** for a rational transfer function)

Proof By definition, for a causal system, $\sigma_0 \in$ROC:

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt = \int_{0}^{\infty} h(t) e^{-st} dt \quad \& \quad \int_{0}^{\infty} |h(t)| e^{-\sigma_0 t} dt < \infty$$

If this converges for σ_0, then consider any $\sigma_1 > \sigma_0$

$$\int_0^{\infty} |h(t)| e^{-\sigma_1 t} dt = \int_0^{\infty} |h(t)| e^{-\sigma_0 t} e^{-(\sigma_1 + \sigma_0)t} dt \le \int_0^{\infty} |h(t)| e^{-\sigma_0 t} dt < \infty$$

so $\sigma_1 \in$ROC

Examples: System Causality

Consider the (LTI 1st order) system with an impulse response

$$h(t) = e^{-t}u(t)$$

This has a transfer function (Laplace transform) and ROC

$$H(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

The transfer function is rational and the ROC is a right half plane. The corresponding **system is causal**.

Consider the system with an impulse response

$$h(t) = e^{-|t|}$$

The system transfer function and ROC

$$H(s) = \int_{-\infty}^{\infty} e^{-|t|}e^{-st}dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt + \int_{-\infty}^{\infty} e^{t}u(-t)e^{-st}dt$$

$$= \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{s^2-1}, \qquad -1 < \mathrm{Re}\{s\} < 1$$

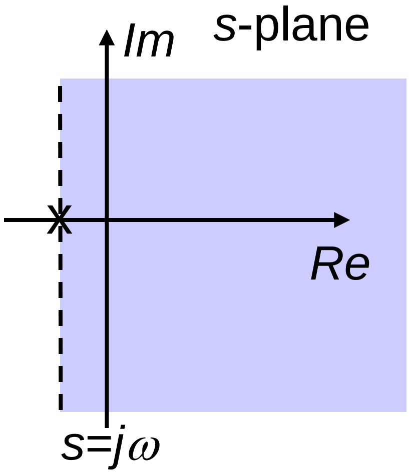
The ROC is **not** the right half plane, so the **system is not causal**

System Stability

Remember, a system is stable if $\forall x : |x| < U \rightarrow |y| < V$, which is equivalent to bounded input signal => bounded output

This is equivalent to saying that an LTI system's impulse is $\int |h(t)| dt < \infty$.

Theorem An LTI system is stable if and only if the ROC of *H*(*s*) includes the entire $j\omega$ axis, i.e. *Re*{*s*} = 0.



Proof The transfer function ROC includes the "axis", $s=j\omega$ along which the Fourier transform has finite energy

Example The following transfer function is stable

$$e^{-at}u(t) \overset{L}{\leftrightarrow} X(s) = \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

Causal System Stability

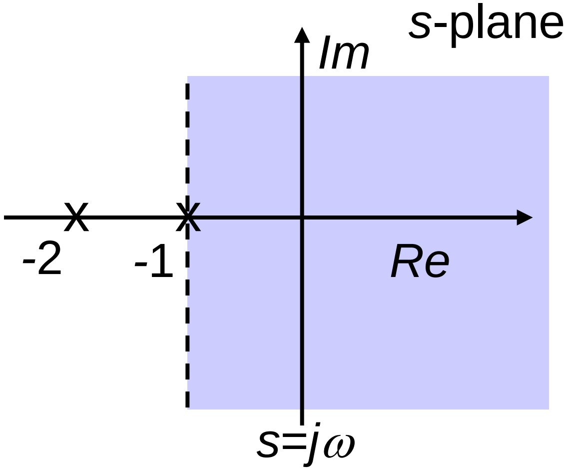
Theorem A causal system with rational system function *H*(*s*) is stable if and only if all of the poles of *H*(*s*) lie in the left-half plane of *s*, i.e. they have negative real parts

Proof Just combine the two previous theorems

Example

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

$$H(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$



Note that the poles of *H*(*s*) correspond to the powers of the exponential response in the time domain. If the real part is negative, they exponential responses decay => stability. Also, the Fourier transform will exist and the imaginary axis lies in the ROC

LTI Differential Equation Systems

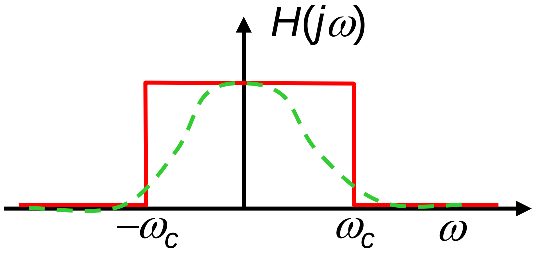
Physical and electrical systems are **causal**

Most physical and electrical systems dissipate energy, they are **stable**. The natural state is “at rest” unless some input/excitation signal is applied to the system

When performing analogue (continuous time) system design, the aim is to produce a time-domain “differential equation” which can then be translated to a known system (electrical circuit …)

This is often done in the frequency domain, which may/may not produce a causal, stable, time-domain differential equation.

Example: low pass filter



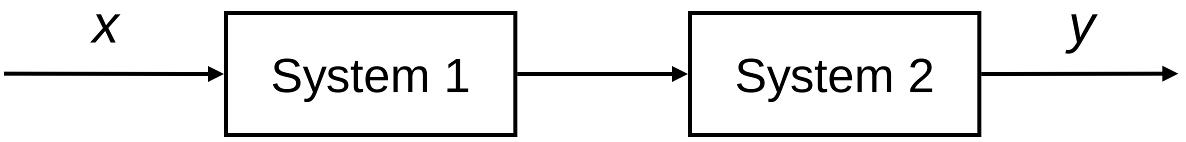
$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$

$$\frac{dh(t)}{dt} + ah(t) = \delta(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

Structures of Sub-Systems

How to combine transfer functions $H_1(s)$ and $H_2(s)$ to get input output transfer function $Y(s) = H(s)X(s)$?

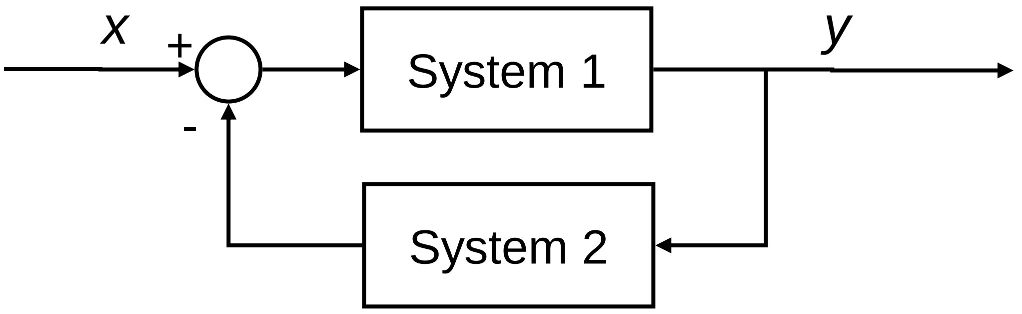
Series/cascade



$$H(s) = H_1(s)H_2(s)$$

Design $H_2()$ to cancel out the effects of $H_1()$

Feedback

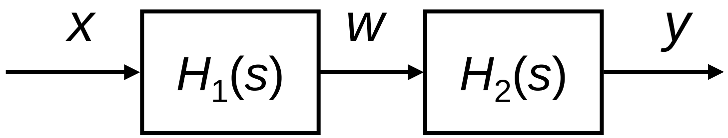


$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Design $H_2()$ to regulate $y(t)$ to $x(t)$, so $H()=1$

Series Cascade & Feedback Proofs

Proof of Series Cascade transfer function

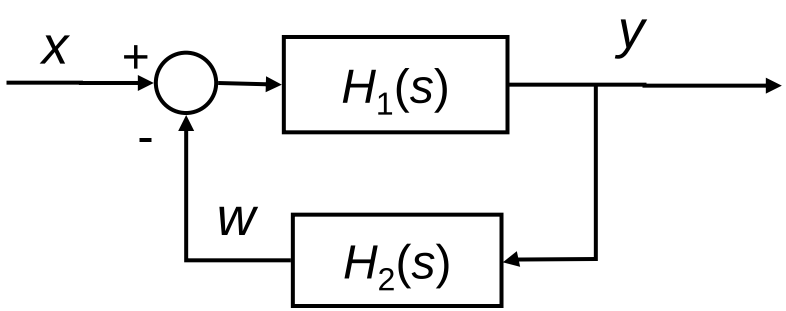


$$H(s) = H_1(s)H_2(s)$$

$$Y(s) = H_2(s)W(s), \qquad W(s) = H_1(s)X(s)$$

$$Y(s) = H_2(s)H_1(s)X(s)$$

Proof of Feedback transfer function



$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

$$W(s) = H_2(s)Y(s), \qquad Y(s) = H_1(s)(X(s) - W(s))$$

$$Y(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s)$$

$$Y(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} X(s)$$

Example: Cascaded 1st Order Systems

Consider two cascaded LTI first order systems

$$H_1(s) = \frac{1}{s+a}$$

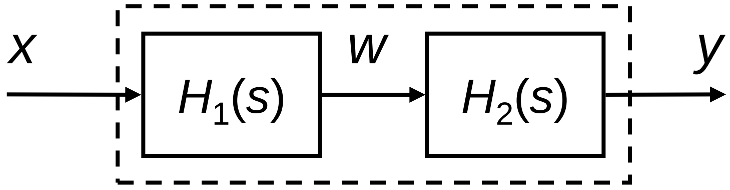
$$H_2(s) = \frac{1}{s+b}$$

$$H(s) = H_1(s)H_2(s)$$

$$= \frac{1}{s+a}\frac{1}{s+b}$$

$$= \frac{1}{s^2+(a+b)s+ab}$$

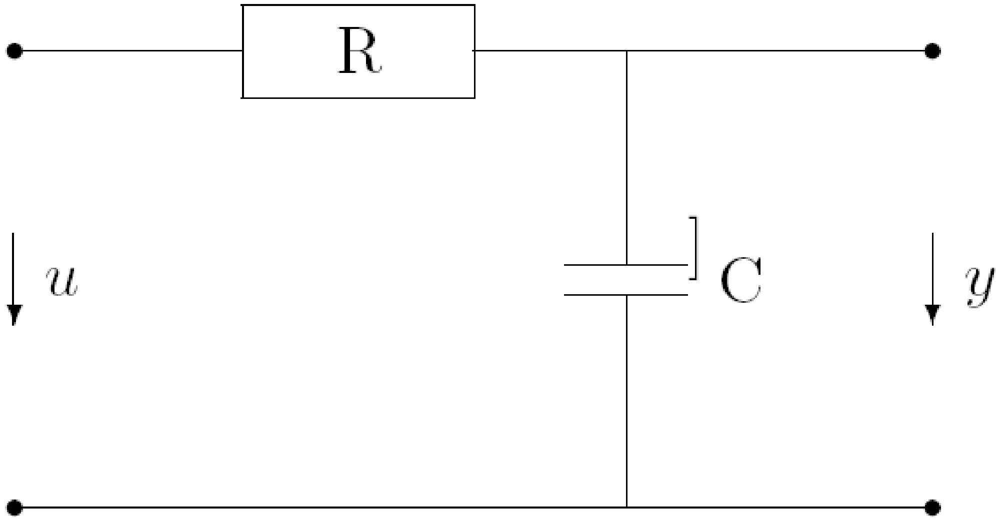
$$h(t) = \frac{1}{(b-a)}\left(e^{-at}-e^{-bt}\right)u(t)$$



$$H(s) = H_1(s)H_2(s)$$

The result of cascading two first order systems is a second order system. However, the roots of this quadratic are purely real (assuming *a* and *b* are real), so the output is not oscillatory, as would be the case with complex roots.

RC Filter as a simple analogue



$$RC\frac{dy}{dt} + y = u$$

Applying the Laplace transform

$$RC[sY(s) - y(0)] + Y(s) = U(s).$$

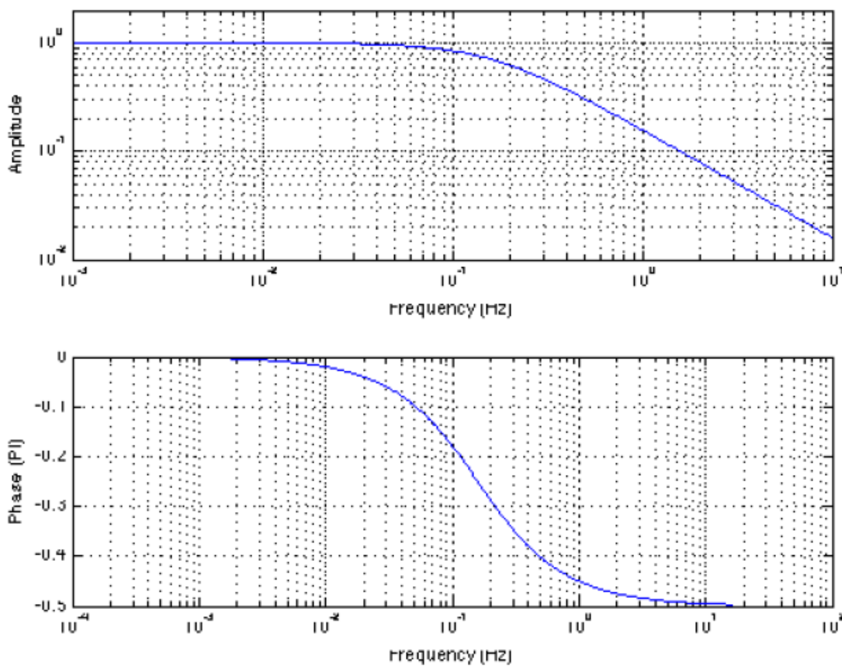
If we assume the intial condition $y(0) = 0$:

$$Y(s)(RCs + 1) = U(s).$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{RCs + 1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Impulse response

… is the inverse transform of the transfer function



$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}}$$

Figure 5.4: Amplitude and phase of the frequency response of an RC–filter with a corner frequency of 0.16 Hz ($RC = 1$).

… time domain …

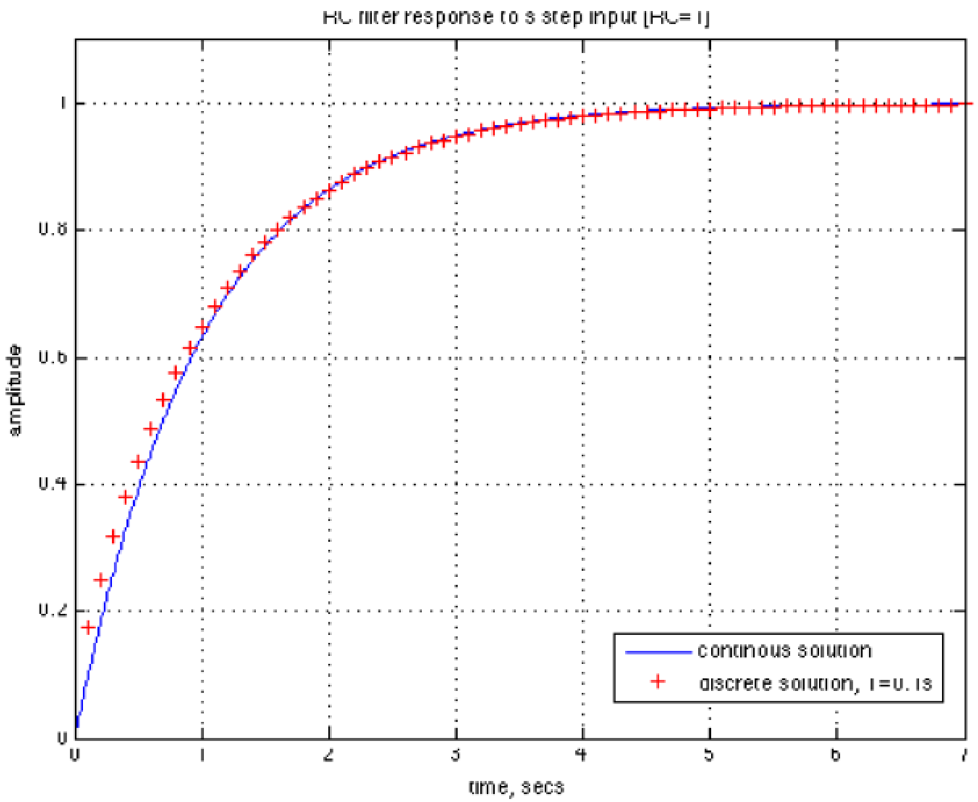


Figure 5.5: Output from an RC–filter for a step function input (RC =
Solid line is the analytic solution, crosses indicate discrete solution w
timestep of $0.1s$