CONVOLUTION PROPERTIES

Scaling Property

- **Scaling property**:

$$x(t) \Longleftrightarrow X(s)$$

$$x(at) \Longleftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right) \quad \text{for } a > 0$$

- Time compression of a signal by a factor a causes expansion of its Laplace transform in s-scale by the same factor.

Time-Convolution & Frequency-Convolution Properties

- **Time-convolution** property:

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$

$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s)$$

- Convolution in time domain is equivalent to multiplication in s (frequency) domain.

- **Frequency-convolution** property:

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$

$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

- **Convolution** in s (frequency) domain is equivalent to **multiplication** in time domain.

Application of the convolution Properties

- Use the time-convolution property of the Laplace transform to determine

$$c(t) = e^{at}u(t) * e^{bt}u(t).$$

- Since $\quad e^{at}u(t) \Leftrightarrow \dfrac{1}{(s-a)} \qquad e^{bt}u(t) \Leftrightarrow \dfrac{1}{(s-b)}$

- Therefore $\quad e^{at}u(t) * e^{bt}u(t) \Leftrightarrow \dfrac{1}{(s-a)(s-b)}$

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

- Perform inverse Laplace transform gives:

$$c(t) = \frac{1}{a-b}(e^{at} - e^{bt})u(t)$$

Relationship with time-domain analysis

- If $h(t)$ is the impulse response of a LTI system, then we have seen in chapter 2 that the system response $y(t)$ to an input $x(t)$ is $x(t)*y(t)$.

- Assuming causality, and that $h(t) \Leftrightarrow H(s)$ and $x(t) \Leftrightarrow X(s)$ then

$$Y(s) = X(s)H(s)$$

- The response $y(t)$ is the zero-state response of the LTI system to the input $x(t)$. It follows that the transfer function $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[\text{zero-state response}]}{\mathcal{L}[\text{input}]}$$

HW5_Ch4: 4.1-1 (a, b, c, d), 4.1-3 (a, b, c, d, f), 4.2-1 (a, b, e, g), 4.2-3 (a,c), 4.2-6

Summary of Laplace Transform Properties (1)

Operation	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$

Summary of Laplace Transform Properties (2)

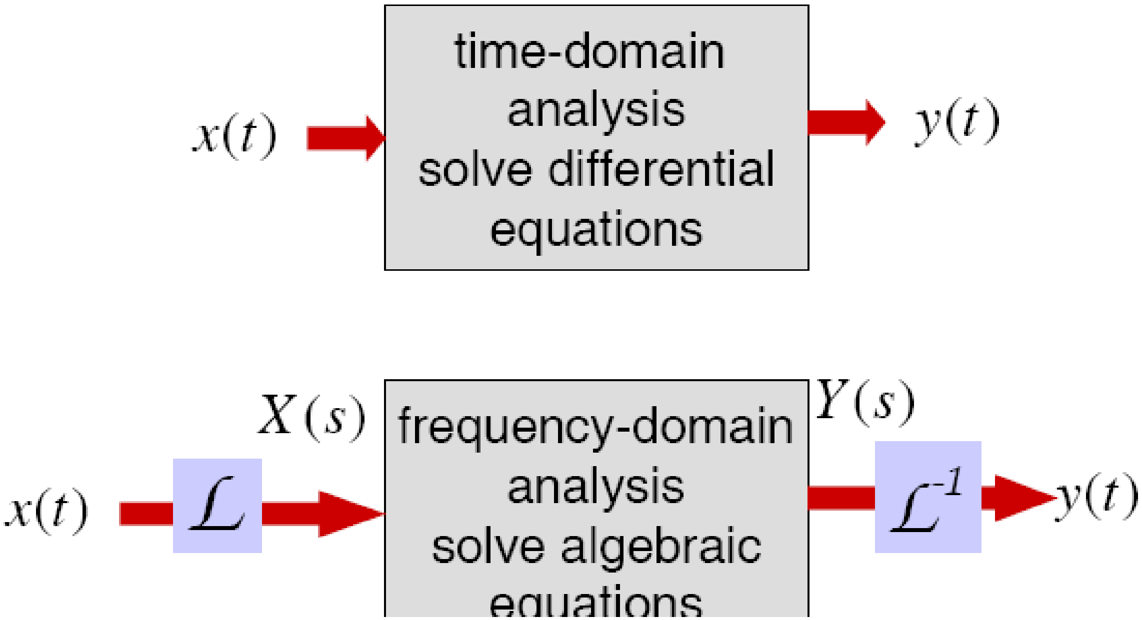
Operation	$x(t)$	$X(s)$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0} \qquad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0 t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\dfrac{dX(s)}{ds}$
Frequency integration	$\dfrac{x(t)}{t}$	$\displaystyle\int_s^{\infty} X(z)\,dz$
Scaling	$x(at),\, a \geq 0$	$\dfrac{1}{a}X\left(\dfrac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\dfrac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \qquad (n > m)$
Final value	$x(\infty)$	$\lim sX(s) \qquad$ [poles of $sX(s)$ in LHP]

Laplace Transform for Solving Differential Equations

- Remember the time-differentiation property of Laplace Transform

$$\frac{d^k y}{dt^k} \Leftrightarrow s^k Y(s)$$

- Exploit this to solve differential equation as algebraic equations:



Example (1)

- Solve the following second-order linear differential equation:

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y(t)=\frac{dx}{dt}+x(t)$$

- Given that $y(0^-)=2,\ \dot{y}(0^-)=1$ and input $x(t)=e^{-4t}u(t).$

Time Domain	Laplace (Frequency) Domain
$\frac{dy}{dt}$	$sY(s)-y(0^-)=sY(s)-2$
$\frac{d^2y}{dt^2}$	$s^2Y(s)-sy(0^-)-\dot{y}(0^-)=s^2Y(s)-2s-1$
$x(t)=e^{-4t}u(t)$	$X(s)=\frac{1}{s+4}$
$\frac{dx}{dt}$	$sX(s)-x(0^-)=\frac{s}{s+4}-0=\frac{s}{s+4}$

Example (2)

Time Domain	Laplace (Frequency) Domain
$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$	$$[s^2Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$
	$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$
	$$(s^2 + 5s + 6)Y(s) = \frac{2s^2 + 20s + 45}{s+4}$$
	$$Y(s) = \frac{2s^2 + 20s + 45}{(s+2)(s+3)(s+4)}$$
$$y(t) = \left(\tfrac{13}{2}e^{-2t} - 3e^{-3t} - \tfrac{3}{2}e^{-4t}\right)u(t)$$	$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$