TIME SHIFTING PROPERTY OF LT

Time Shifting Property of the Laplace transform

- **Time Shifting** property:

$$x(t) \Longleftrightarrow X(s)$$
$$x(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad \text{for } t_0 \geq 0$$

- Delaying $x(t)$ by t_0 (i.e. time shifting) amounts to multiplying its transform $X(s)$ by e^{-st_0}.

- Remember that $x(t)$ starts at $t = 0$, and $x(t - t_0)$ starts at $t = t_0$.

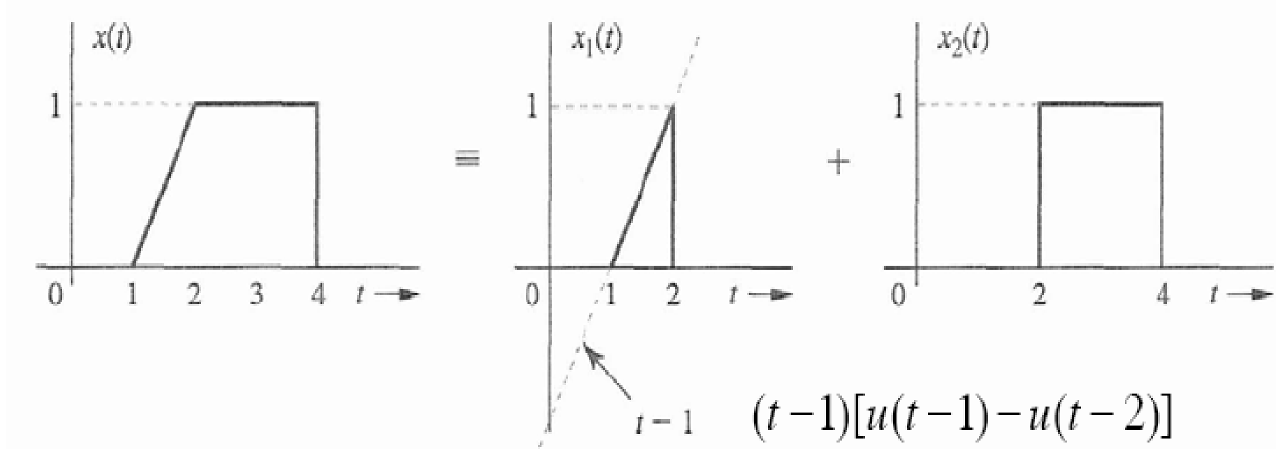
- Therefore, the more accurate statement of the time shifting property is:

$$x(t)u(t) \Longleftrightarrow X(s)$$
$$x(t-t_0)u(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad t_0 \geq 0$$

Application of Time Shifting

- Find the Laplace transform of $x(t)$ as shown:

$[u(t-2)-u(t-4)]$



$(t-1)[u(t-1)-u(t-2)]$

$$x(t)=(t-1)[u(t-1)-u(t-2)] \quad + \quad [u(t-2)-u(t-4)]$$

$$=(t-1)u(t-1)-\underline{(t-1)u(t-2) \quad + \quad u(t-2)}-u(t-4)$$

$$=(t-1)u(t-1) \quad - \quad (t-2)u(t-2) \quad - \quad u(t-4)$$

$$u(t) \Longleftrightarrow \frac{1}{s}$$

$$tu(t) \Longleftrightarrow 1/s^2$$

Time shift

$$(t-2)u(t-2) \Longleftrightarrow \frac{1}{s^2}e^{-2s}$$

$$(t-1)u(t-1) \Longleftrightarrow \frac{1}{s^2}e^{-s}$$

$$u(t-4) \Longleftrightarrow \frac{1}{s}e^{-4s}$$

$$X(s)=\frac{1}{s^2}e^{-s}-\frac{1}{s^2}e^{-2s}-\frac{1}{s}e^{-4s}$$

Frequency Shifting Property

- **Frequency Shifting** property:

$$x(t) \Longleftrightarrow X(s)$$

$$x(t)e^{s_0 t} \Longleftrightarrow X(s - s_0)$$

- Frequency shifting the transform $X(s)$ by s_0 amounts to multiplying its time signal by $e^{s_0 t}$.

- Observe symmetry (or duality) between frequency-shift and time-shift properties.

$$x(t) \Longleftrightarrow X(s)$$

$$x(t - t_0) \Longleftrightarrow X(s)e^{-st_0} \quad \text{for } t_0 \geq 0$$

Application of Frequency Shifting

- Given $\cos bt\, u(t) \Longleftrightarrow \dfrac{s}{s^2+b^2}$, show that $e^{-at}\cos bt\, u(t) \Longleftrightarrow \dfrac{s+a}{(s+a)^2+b^2}$.

- Apply frequency-shifting property with frequency shift $s_0 = -a$.

- Replace s with (s+a) means frequency shift by -a. This yields the RHS of the equation. By frequency-shifting property, we need to multiply the LHS by e^{-at}.

Time-Differentiation Property

- **Time-differentiation** property:

$$x(t) \Longleftrightarrow X(s)$$
$$\frac{dx}{dt} \Longleftrightarrow sX(s) - x(0^-)$$

- Repeated application of this property yields:

$$\frac{d^2x}{dt^2} \Longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-)$$
$$\frac{d^nx}{dt^n} \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$
$$= s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$$

where $x^{(r)}(0^-)$ is d^rx/dt^r at $t = 0^-$.

- Frequency-differentiation property:

$$x(t) \Longleftrightarrow X(s)$$
$$tx(t) \Longleftrightarrow -\frac{d}{ds}X(s)$$

Proof of Time-Differentiation Property

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\, dt$$

- Integration by parts gives: $u = e^{-st}$ $dv = dx$

$$\mathcal{L}\left[\frac{dx}{dt}\right] = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\, dt$$

- For the Laplace integral to converge, it is necessary that

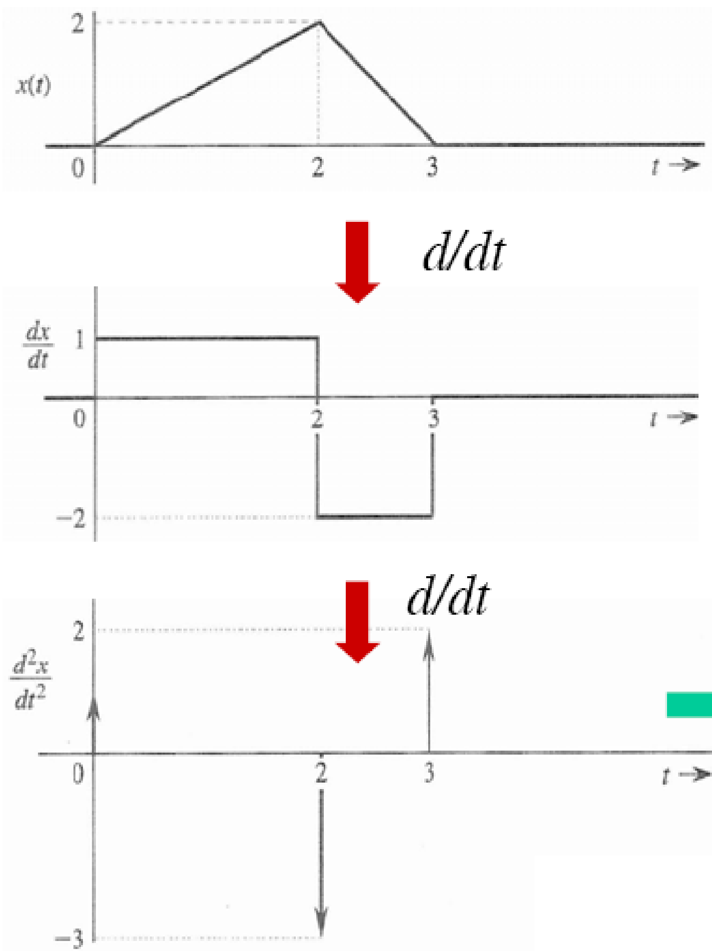
$$x(t)e^{-st} \rightarrow 0 \text{ as } t \rightarrow \infty$$

- Therefore we get:

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -x(0^-) + sX(s)$$

Application of Time-Differentiation

- Find the Laplace transform of the signal x(t) using time differentiation and time-shifting properties.



$$X(s) = \frac{1}{s^2}\left(1 - 3e^{-2s} + 2e^{-3s}\right)$$

$$s^2 X(s) - 0 - 0 = 1 - 3e^{-2s} + 2e^{-3s}$$

$$x(0^-) = \dot{x}(0^-) = 0, \text{ and } \delta(t) \Longleftrightarrow 1$$

$$\frac{d^2x}{dt^2} \Longleftrightarrow s^2 X(s) - s x(0^-) - \dot{x}(0^-)$$

Time-differentiation & time shifting properties

$$\frac{d^2x}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = \mathcal{L}\left[\delta(t) - 3\delta(t-2) + 2\delta(t-3)\right]$$

Time-Integration Property

- **Time-integration** property:

$$x(t) \Longleftrightarrow X(s)$$
$$\int_{0^-}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(s)}{s}$$

- The dual property of time-integration is the frequency-integration property:

$$x(t) \Longleftrightarrow X(s)$$
$$\frac{x(t)}{t} \Longleftrightarrow \int_{s}^{\infty} X(z)\,dz$$