Continuous-Time System Analysis Using The Laplace Transform

Outline

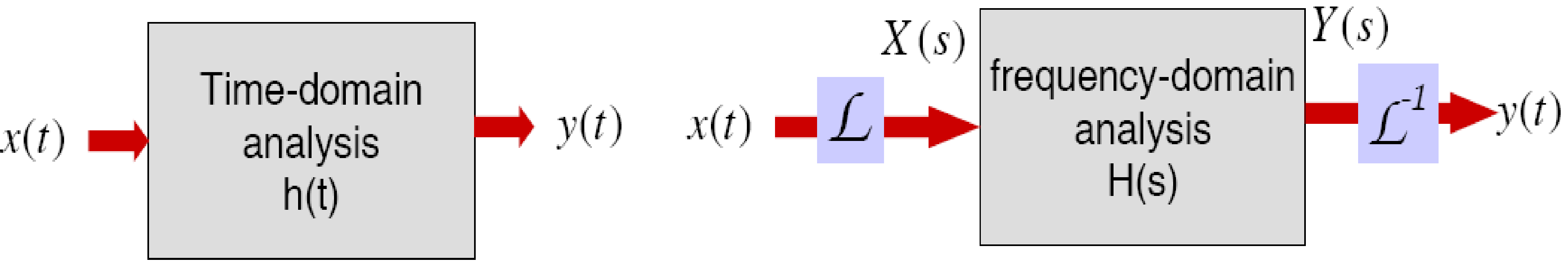
- Introduction
- Properties of Laplace Transform
- Solution of Differential Equations
- Analysis of Electrical Networks
- Block Diagrams and System Realization
- Frequency Response of an LTIC System
- Filter Design by Placement of Poles and Zeros of H(s)

Why Laplace Transform?

- Laplace transform is the **dual** (or complement) of the time-domain analysis.
- In **time-domain** analysis, we break input x(t) into **impulsive component**, and sum the system response to all these components.
- In **frequency-domain** analysis, we break the input x(t) into **exponentials components** of the form e^{st}, where s is the complex frequency:

$$s = \alpha + j\omega$$

- Laplace transform is the tool to map signals and system behaviour from the time-domain into the frequency domain.



Definition of Two-sided Laplace Transform

- For a signal *x(t)*, its **Laplace transform is defined by:**

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

- The signal *x(t)* is said to be the **inverse Laplace transform** of X(s). It can be shown that

$$x(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

  where c is a constant chosen to ensure the **convergence** of the first integral.

- This general definite is known as **two-sided** (or **bilateral**) Laplace Transform.

$$x(t) = 3e^{-5t} \xrightarrow{LT} X(s) = \frac{3}{s+5}$$

$$\left|X(s)\right| = \frac{3}{\sqrt{(\sigma+5)^2 + \omega^2}}$$

$$x(t) \approx \sum_{i=-\infty}^{\infty} X(s_i)\, e^{s_i t}$$

Definition of One-sided Laplace Transform

- Let us assume that **all signals are causal**.
  For this the Laplace transform is defined as:

$$X(s) = \mathcal{L}[x(t)] = \int_0^{\infty} x(t) e^{-st} dt$$

- This is known as **one-side** (or **unilateral**) Laplace transform.

- Remember that the Laplace transform is **a linear tranform**

$$\mathcal{L}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 \mathcal{L}\{f_1(t)\} + k_2 \mathcal{L}\{f_2(t)\}$$

A few examples

- Find the Laplace transform of δ(t) and u(t).

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1 \qquad \text{for all } s$$

$$\boxed{\mathcal{L}[\delta(t)] \quad \Leftrightarrow \quad 1}$$

$$\mathcal{L}[u(t)] = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad \operatorname{Re} s > 0$$

$$\boxed{\mathcal{L}[u(t)] \quad \Leftrightarrow \quad \frac{1}{s}}$$

A few examples (2)

- Find the Laplace transform of $e^{at}u(t)$ and $\cos \omega_0 t\, u(t)$.

$$\mathcal{L}[e^{at}u(t)] = \int_0^{\infty} e^{at}e^{-st}\,dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

$$\mathcal{L}[e^{at}u(t)] \Leftrightarrow \frac{1}{s-a}$$

$$\mathcal{L}[\cos \omega_0 t\, u(t)] = \frac{1}{2}\mathcal{L}[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{s}{s^2+\omega_0^2}$$

$$\mathcal{L}[\cos \omega_0 t u(t)] \Leftrightarrow \frac{s}{s^2+\omega_0^2}$$

Laplace transform Pairs (1)

- Finding inverse Laplace transform requires integration in the complex plane – beyond scope of this course.
- So, use a Laplace transform table (analogous to the convolution table).

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$

Laplace transform Pairs (2)

No.	$x(t)$	$X(s)$
5	$e^{\lambda t} u(t)$	$\dfrac{1}{s-\lambda}$
6	$te^{\lambda t} u(t)$	$\dfrac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t} u(t)$	$\dfrac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\dfrac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\dfrac{b}{s^2+b^2}$
9a	$e^{-at} \cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at} \sin bt\, u(t)$	$\dfrac{b}{(s+a)^2+b^2}$

Laplace transform Pairs (3)

No.	$x(t)$	$X(s)$
10a	$re^{-at}\cos(bt+\theta)\,u(t)$	$\dfrac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\,u(t)$	$\dfrac{0.5re^{j\theta}}{s+a-jb}+\dfrac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\,u(t)$ $r=\sqrt{\dfrac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta=\tan^{-1}\left(\dfrac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b=\sqrt{c-a^2}$	$\dfrac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt+\dfrac{B-Aa}{b}\sin bt\right]u(t)$ $b=\sqrt{c-a^2}$	$\dfrac{As+B}{s^2+2as+c}$

Examples of Inverse Laplace Transform (1)

- Finding inverse Laplace transform of $\dfrac{7s-6}{s^2-s-6}$. (use partial fraction)

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

- To find k_1 which corresponds to the term (s+2), cover up (s+2) in X(s), and substitute s = -2 (i.e. s+2=0) in the remaining expression:

$$k_1 = \left.\frac{7s-6}{(s+2)(s-3)}\right|_{s=-2} = \frac{-14-6}{-2-3} = 4$$

- Similarly for k_2:

$$k_2 = \left.\frac{7s-6}{(s+2)(s-3)}\right|_{s=3} = \frac{21-6}{3+2} = 3$$

- Therefore

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{3}{s-3}$$

Examples of Inverse Laplace Transform (2)

- Easy to make mistake with partial fraction.
- Method to check correctness of:

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{3}{s-3}$$

- Substitute s = 0 into the equation (could use other values, but this is most convenient):

$$X(0) = \frac{-6}{(+2)(-3)} = 1 = \frac{4}{2} + \frac{3}{-3}$$

- Therefore, using Pair 5 from table:

$$x(t) = \mathcal{L}^{-1}\left(\frac{4}{s+2} + \frac{3}{s-3}\right) = (4e^{-2t} + 3e^{3t})u(t)$$

Examples of Inverse Laplace Transform (3)

- Finding the inverse Laplace transform of $\dfrac{2s^2-5}{(s+1)(s+2)}$.

- The partial fraction of this expression is less straight forward. If the power of numerator polynomial (M) is the same as that of denominator polynomial (N), we need to add the coefficient of the highest power in the numerator to the normal partial fraction form:

$$X(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

- Solve for k_1 and k_2 via "covering":

$$k_1 = \left.\frac{2s^2+5}{(s+1)(s+2)}\right|_{s=-1} = \frac{2+5}{-1+2} = 7$$

$$k_2 = \left.\frac{2s^2+5}{(s+1)(s+2)}\right|_{s=-2} = \frac{8+5}{-2+1} = -13$$

- Therefore $X(s) = 2 + \dfrac{7}{s+1} - \dfrac{13}{s+2}$

- Using pairs 1 & 5:

$$x(t) = 2\delta(t) + (7e^{-t} - 13e^{-2t})u(t)$$