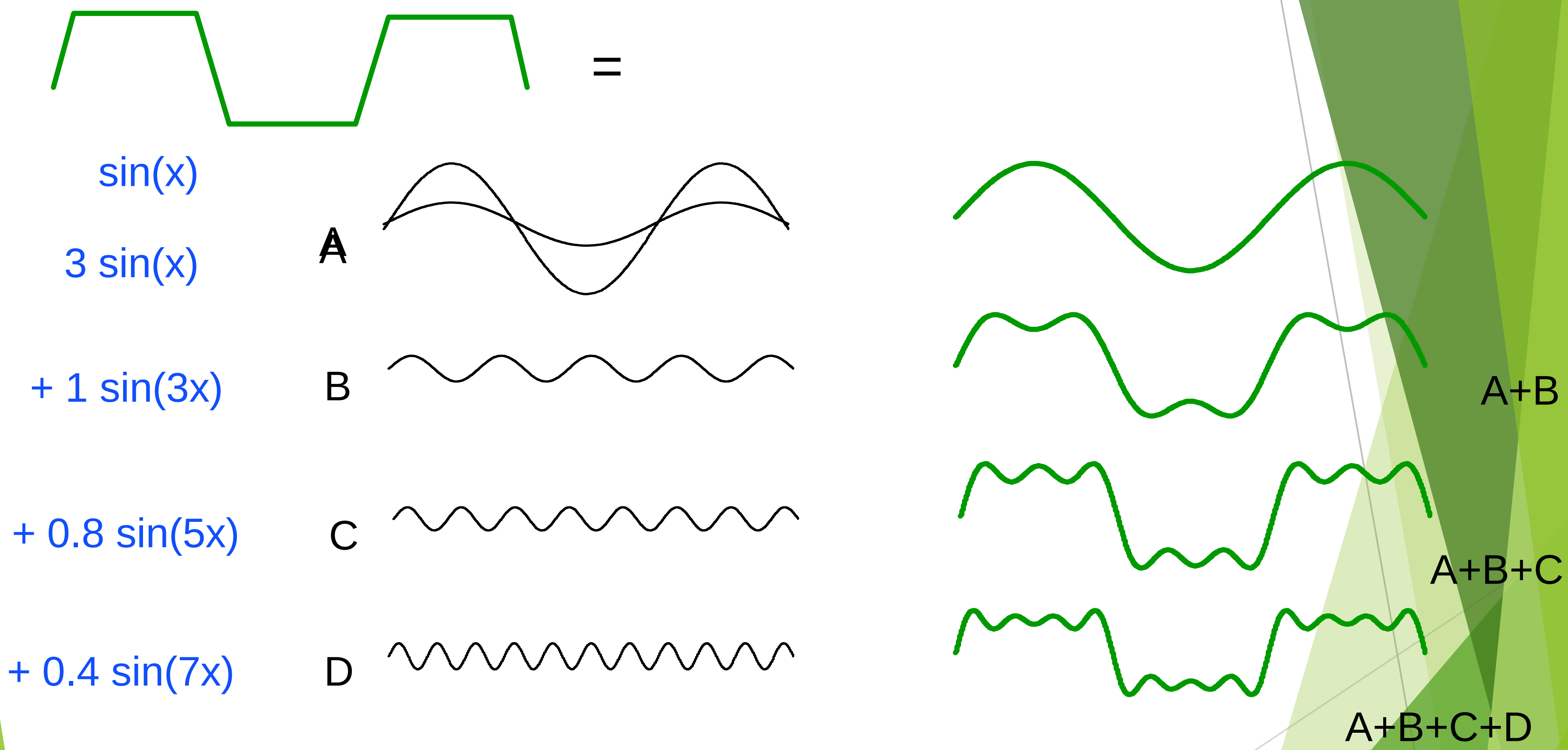
Review on Fourier Transform & Magnitude and Phase Spectrum

Fourier Transform and Spectra

Topics:

- Fourier (FT) and Inverse Fourier Transform
- Properties of Fourier Transforms & Some Useful Theorems
- Parseval's Theorem and Energy Spectral Density
- Magnitude and Phase Spectra
  - Dirac Delta Function and Unit Step Function
  - Rectangular and Triangular Pulses

A sum of sines and cosines



Fourier transform of a waveform

- **Definition: Fourier transform**

The *Fourier transform* (FT) of a waveform $w(t)$ is:

$$W(f) = \Im[w(t)] = \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi ft} dt$$

where

$\Im[.]$ denotes the Fourier transform of [.]

f is the frequency parameter with units of Hz (i.e., 1/s).

- $W(f)$ is also called a two-sided spectrum of $w(t)$, since both positive and negative frequency components are obtained from the definition

Evaluation Techniques for FT Integral

- One of the following techniques can be used to evaluate a FT integral:
  - Direct integration.
  - Tables of Fourier transforms or Laplace transforms.
  - FT theorems.
  - Superposition to break the problem into two or more simple problems.
  - Differentiation or integration of $w(t)$.
  - Numerical integration of the FT integral on the PC via MATLAB
  - Fast Fourier transform (FFT)

Fourier transform of a waveform

- **Definition: Inverse Fourier transform**

  The *Inverse Fourier transform* (FT) of a waveform $w(t)$ is:

$$w(t) = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df$$

- The functions $w(t)$ and $W(f)$ constitute a ***Fourier transform pair***.

$$w(t) = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df \quad \Longleftrightarrow \quad W(f) = \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi ft} dt$$

Time Domain description (Inverse FT)

Frequency Domain description (FT)

Fourier transform - Sufficient conditions

- The waveform $w(t)$ is Fourier transformable if it satisfies both **Dirichlet conditions:**
  1) Over any time interval of finite length, the function $w(t)$ is single valued with a finite number of maxima and minima, and the number of discontinuities (if any) is finite.
  2) $w(t)$ is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |w(t)|\, dt < \infty$$

- Above conditions are sufficient, but not necessary.
- A weaker sufficient condition for the existence of the Fourier transform is:

$$E = \int_{-\infty}^{\infty} |w(t)|^2\, dt < \infty$$

Finite Energy

- where E is the normalized energy.
- This is the finite-energy condition that is satisfied by all physically realizable waveforms.
- Conclusion: All physical waveforms encountered in engineering practice are Fourier transformable.

Spectrum of an exponential pulse

Let $w(t)$ be a decaying exponential pulse that is switched on at $t = 0$. That is,

$$w(t) = \begin{cases} e^{-t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Directly integrating the FT integral, we get

$$W(f) = \int_0^{\infty} e^{-t} e^{-j2\pi f t}\, dt = \left. \frac{e^{-(1+j2\pi f)t}}{1 + j2\pi f} \right|_0^{\infty}$$

or

$$W(f) = \frac{1}{1 + j2\pi f}$$

In other words, the FT pair is

$$\begin{Bmatrix} e^{-t}, & t > 0 \\ 0, & t < 0 \end{Bmatrix} \leftrightarrow \frac{1}{1 + j2\pi f}$$

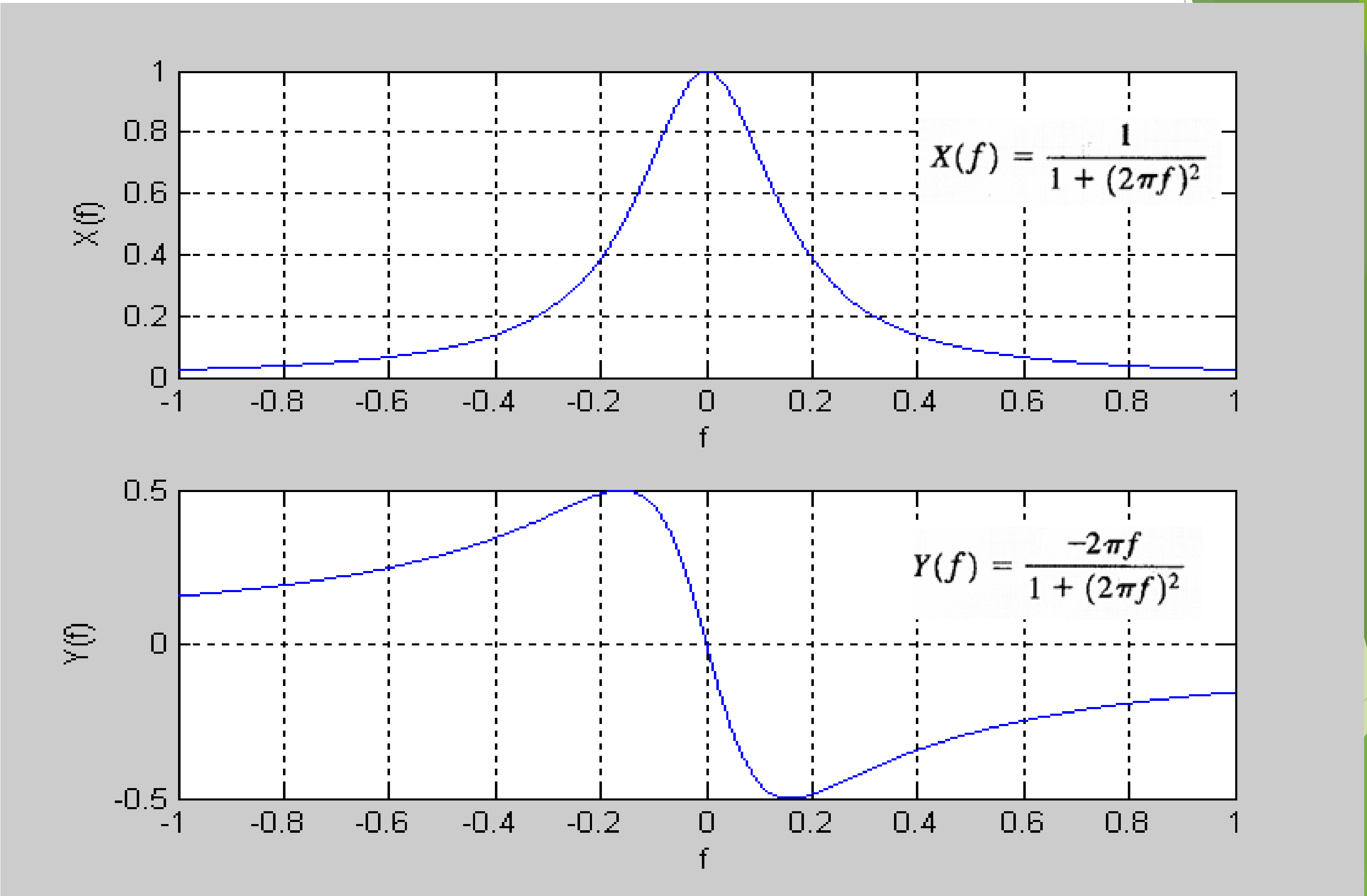
The spectrum can also be expressed in terms of the quadrature functions by rationalizing the denominator of Eq. (2–34); thus,

$$X(f) = \frac{1}{1 + (2\pi f)^2} \quad \text{and} \quad Y(f) = \frac{-2\pi f}{1 + (2\pi f)^2}$$

The magnitude–phase form is

$$|W(f)| = \sqrt{\frac{1}{1 + (2\pi f)^2}} \quad \text{and} \quad \theta(f) = -\tan^{-1}(2\pi f)$$

Plots of functions X(f) and Y(f)



Theorems on Fourier Transforms

- **Theorem :** *Spectral symmetry of real signals*

  If w(t) is real, then

$$W(-f) = W^*(f)$$

Superscript asterisk denotes the conjugate operation.

- **Proof:** $W(f) = \Im[w(t)] = \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi ft} dt$

Substitute *-f*

Take the conjugate

$$W(-f) = \int_{-\infty}^{\infty} w(t) e^{j2\pi ft} dt \quad = \quad W^*(f) = \int_{-\infty}^{\infty} w^*(t) e^{j2\pi ft} dt$$

Since *w*(*t*) is real, ***w**(*t*) = *w*(*t*),** and it follows that ***W*(-*f*) = *W**(*f*).**

•If ***w*(*t*)** is real and is an even function of *t*, ***W*(*f*)** is real.

•If ***w*(*t*)** is real and is an odd function of *t*, ***W*(*f*)** is imaginary.

Theorems on Fourier Transforms

Corollaries of $W(-f) = W^*(f)$

If ***w*(*t*)** *is* real,

- Magnitude spectrum is even about the origin (i.e., $f = 0$),

  $|W(-f)| = |W(f)|$ ....(*A*)

- Phase spectrum is odd about the origin.

  $\theta(-f) = -\theta(f)$ ....(*B*)

Proof. $W(f) = |W(f)|e^{j\theta(f)}$

Then $W(-f) = |W(-f)|e^{j\theta(-f)}$ and

$$W^*(f) = |W(f)|e^{-j\theta(f)}$$

$$|W(-f)|e^{j\theta(-f)} = |W(f)|e^{-j\theta(f)}$$

Since, ***W*(-*f*) = *W**(*f*)**
We see that corollaries (A) and (B) are true.

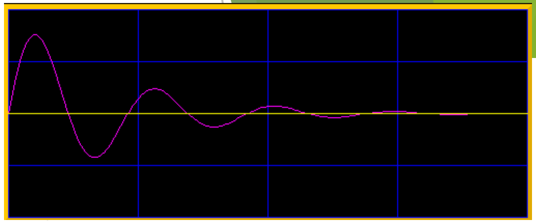
Theorems on Fourier Transforms - Summary

- *f*, called frequency and having units of hertz, is just a parameter of the FT that specifies what frequency we are interested in looking for in the waveform $w(t)$.
- The FT looks for the frequency *f* in the $w(t)$ over *all* time, that is, over $-\infty < t < \infty$
- $W(f)$ can be complex, even though $w(t)$ is real.
- If $w(t)$ is real, then $W(-f) = W^*(f)$.

TABIE 2-1: SOME FOURIER TRANSFORM THEOREMS

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t) + a_2 w_2(t)$	$a_1 W_1(f) + a_2 W_2(f)$
Time delay	$w(t - T_d)$	$W(f)\, e^{-j\omega T_d}$
Scale change	$w(at)$	$\dfrac{1}{\lvert a \rvert} W\left(\dfrac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation [$w(t)$ is real]	$w(t)\cos(w_c t + \theta)$	$\frac{1}{2}[e^{j\theta} W(f - f_c) + e^{-j\theta} W(f + f_c)]$
Complex signal frequency translation	$w(t)\, e^{j\omega_c t}$	$W(f - f_c)$
Bandpass signal	$\mathrm{Re}\{g(t)\, e^{j\omega_c t}\}$	$\frac{1}{2}[G(f - f_c) + G^*(-f - f_c)]$
Differentiation	$\dfrac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$
Integration	$\displaystyle\int_{-\infty}^{t} w(\lambda) d\lambda$	$(j2\pi f)^{-1} W(f) + \frac{1}{2} W(0)\, \delta(f)$
Convolution	$w_1(t) * w_2(t) = \displaystyle\int_{-\infty}^{\infty} w_1(\lambda) \cdot w_2(t - \lambda)\, d\lambda$	$W_1(f) W_2(f)$
Multiplication[b]	$w_1(t) w_2(t)$	$W_1(f) * W_2(f) = \displaystyle\int_{-\infty}^{\infty} W_1(\lambda)\, W_2(f - \lambda)\, d\lambda$
Multiplication	$t^n w(t)$	$(-j2\pi)^{-n} \dfrac{d^n W(f)}{df^n}$

Example 2-3: Spectrum of a Damped Sinusoid



Let the damped sinusoid be given by

$$w(t) = \begin{cases} e^{-t/T} \sin \omega_0 t, & t > 0,\ T > 0 \\ 0, & t < 0 \end{cases}$$

The spectrum of this waveform is obtained by evaluating the FT. This is easily accomplished by using the result of the previous example plus some of the Fourier theorems. From Eq. (2–34), and using the scale-change theorem of Table 2–1, where $a = 1/T$, we find that

$$\left\{ \begin{matrix} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{matrix} \right\} \leftrightarrow \frac{T}{1 + j(2\pi f T)}$$

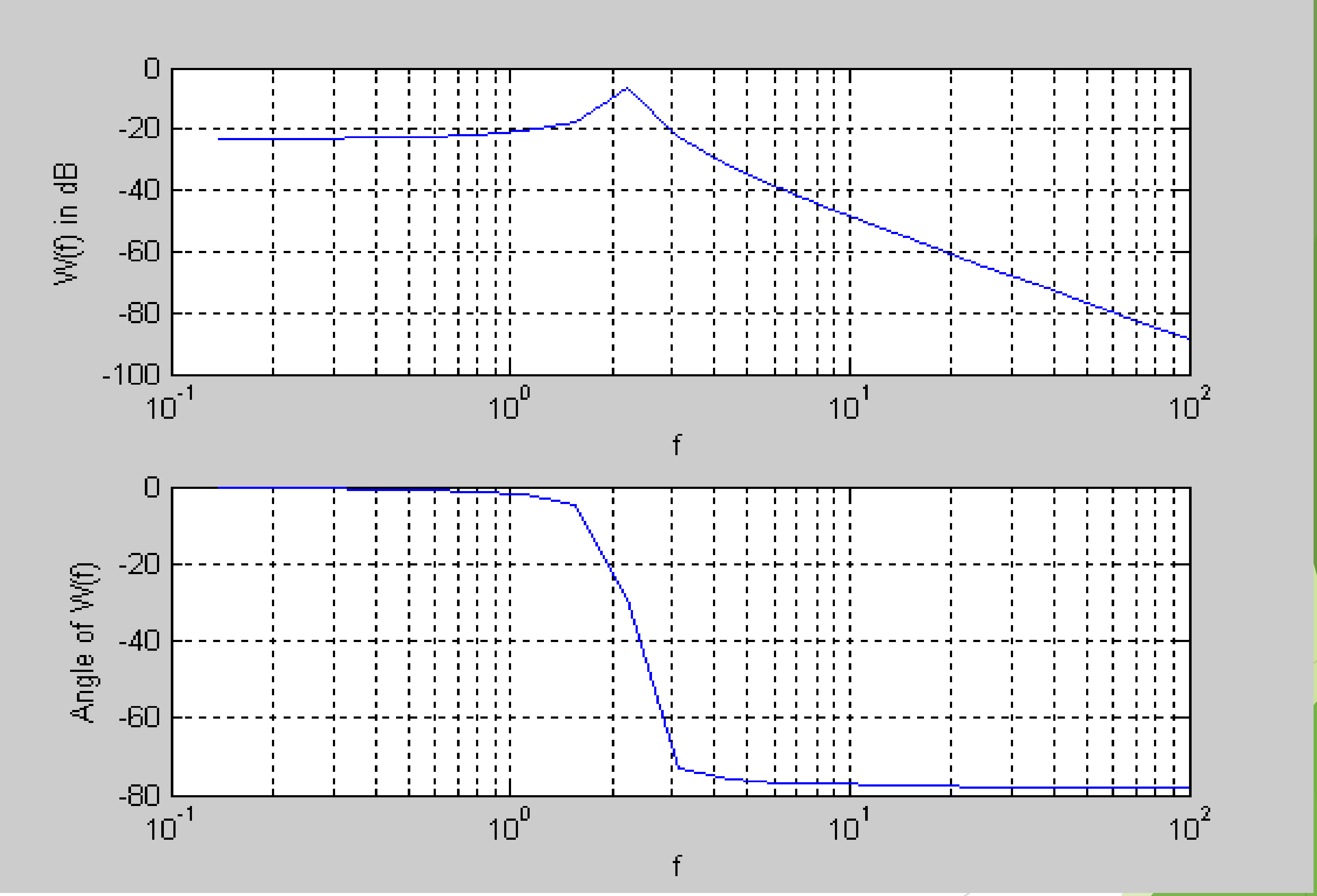
Using the real signal frequency translation theorem with $\theta = -\pi/2$, we get

$$W(f) = \frac{1}{2}\left\{ e^{-j\pi/2} \frac{T}{1 + j2\pi T(f - f_0)} + e^{j\pi/2} \frac{T}{1 + j2\pi T(f + f_0)} \right\}$$

$$= \frac{T}{2j}\left\{ \frac{1}{1 + j2\pi T(f - f_0)} - \frac{1}{1 + j2\pi T(f + f_0)} \right\} \quad (2\text{–}44)$$

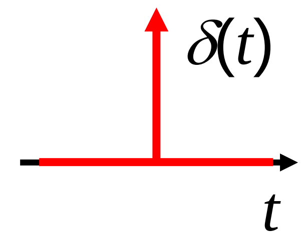
- Spectral Peaks of the Magnitude spectrum has moved to $f=f_o$ and $f=-f_o$ due to multiplication with the sinusoidal.

Example 2-3: Variation of W(*f*) with *f*



Dirac Delta Function & Unit Step Function

- **Definition:**
- The ***Dirac delta*** *function δ(x)* is defined



$$\int_{-\infty}^{\infty} w(x)\delta(x)dx = w(0) \quad (2\text{-}45)$$

where w(x) is any function that is continuous at x = 0.
An alternative definition of δ(x) is:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \quad (2\text{-}46a)$$

and

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \quad (2\text{-}46b)$$

From (2-45), the Sifting Property of the δ function is

$$\int_{-\infty}^{\infty} w(x)\delta(x - x_0)dx = w(x_0) \quad (2\text{-}47)$$

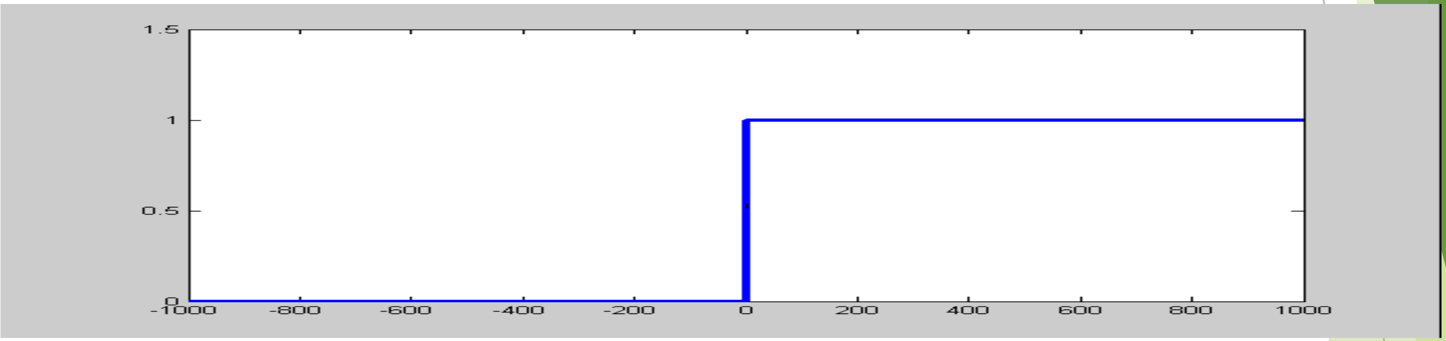
If δ(x) is an even function the integral of the δ function is given by:

$$\delta(x) = \int_{-\infty}^{\infty} e^{\pm j2\pi xy} dy$$

Unit Step Function

- **Definition:** The Unit Step function u(t) is:

$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases} \qquad (2\text{-}49)$$



Because δ(λ) is zero, except at λ = 0, the Dirac delta function is related to the unit step function by

$$\frac{du(t)}{dt}=\delta(t) \qquad (2\text{-}51)$$

and consequently,

$$\int_{-\infty}^{t}\delta(\lambda)d\lambda=u(t) \qquad (2\text{-}50)$$

Example 2-4: Spectrum of a Sine Wave

$$v(t) = A \sin \omega_0 t \quad \text{where} \quad \omega_0 = 2\pi f_0$$

From Eq. (2–26), the spectrum is

$$V(f) = \int_{-\infty}^{\infty} A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega t}\, dt$$

$$\delta(x) = \int_{-\infty}^{\infty} e^{\pm j 2\pi x y}\, dy$$

$$= \frac{A}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\, dt - \frac{A}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\, dt$$

By Eq. (2–48), these integrals are equivalent to Dirac delta functions. That is,

$$V(f) = j\frac{A}{2}\left[\delta(f+f_0) - \delta(f-f_0)\right]$$

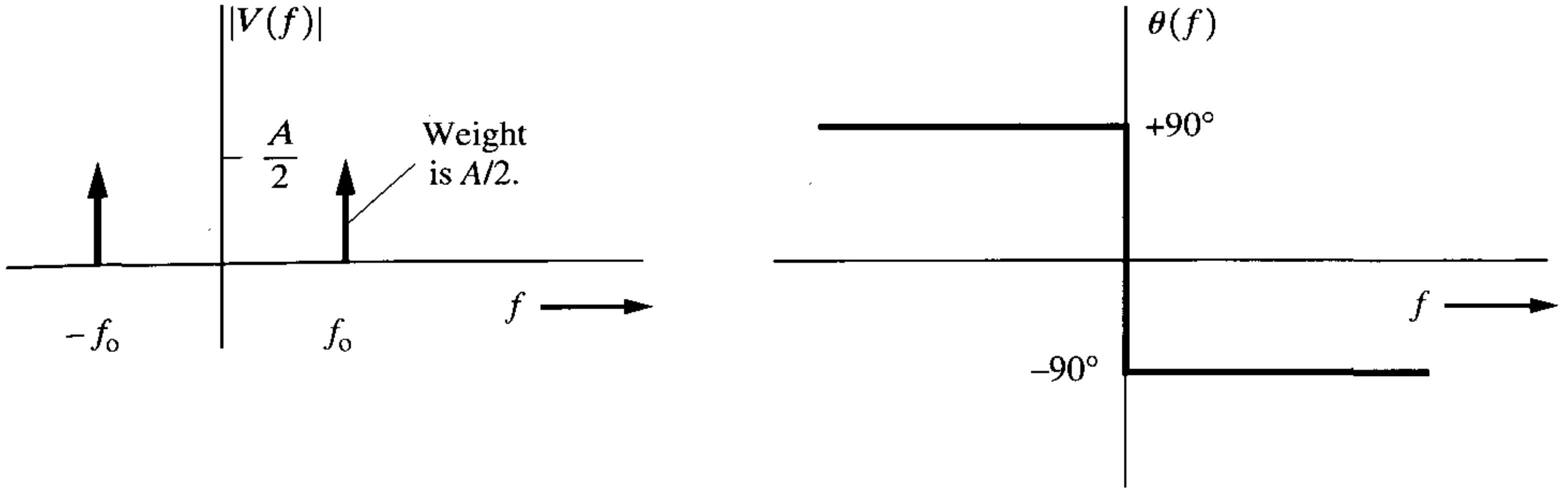
Note that this spectrum is imaginary, as expected, because $v(t)$ is real and odd. In addition, a meaningful expression was obtained for the Fourier transform, although $v(t)$ was of the infinite energy type and not absolutely integrable. That is, this $v(t)$ does not satisfy the sufficient (but not necessary) Dirichlet conditions as given by Eqs. (2–31) and (2–32).

The magnitude spectrum is

$$|V(f)| = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$$

Example 2-4: Spectrum of a Sine Wave (contd..)

$$\theta(f) = \begin{cases} -\pi/2, & f > 0 \\ +\pi/2, & f < 0 \end{cases} \text{ radians} = \begin{cases} -90°, & f > 0 \\ 90°, & f < 0 \end{cases}$$



(a) Magnitude Spectrum

(b) Phase Spectrum ($\theta_0 = 0$)

Now let us generalize the sinusoidal waveform to one with an arbitrary phase angle θ_0. Then

$$w(t) = A \sin(\omega_0 t + \theta_0) = A \sin[\omega_0(t + \theta_0/\omega_0)]$$

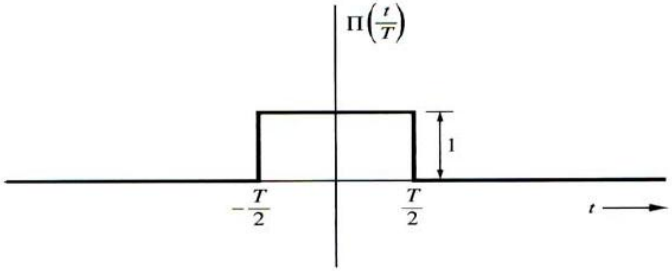
and, by using the time delay theorem, the spectrum becomes

$$W(f) = j\frac{A}{2} e^{j\theta_0(f/f_0)} [\delta(f + f_0) - \delta(f - f_0)]$$

Rectangular and Triangular Pulses

DEFINITION. Let $\Pi(\cdot)$ denote a single *rectangular pulse*. Then

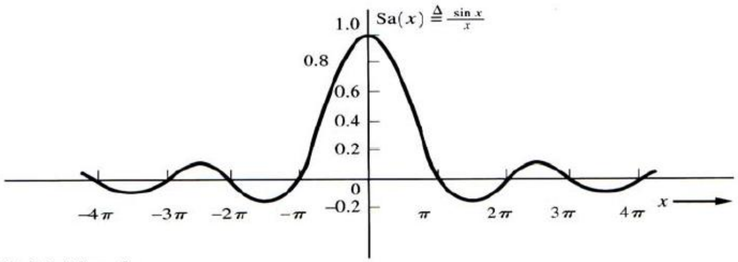
$$\Pi\left(\frac{t}{T}\right) \triangleq \begin{cases} 1, & |t| \le \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$



(a) Rectangular Pulse

DEFINITION. Sa(·) denotes the function†

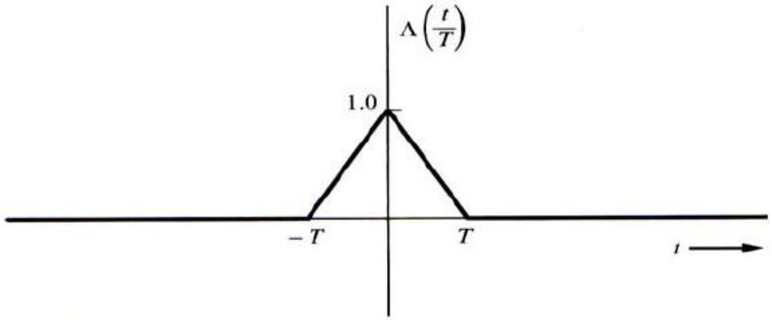
$$Sa(x) = \frac{\sin x}{x}$$



(b) Sa(x) Function

DEFINITION. Let $\Lambda(\cdot)$ denote the triangular function. Then

$$\Lambda\left(\frac{t}{T}\right) \triangleq \begin{cases} 1 - \frac{|t|}{T}, & |t| \le T \\ 0, & |t| > T \end{cases}$$



Triangular Function

Example 2-5: Spectrum of a Rectangular Pulse

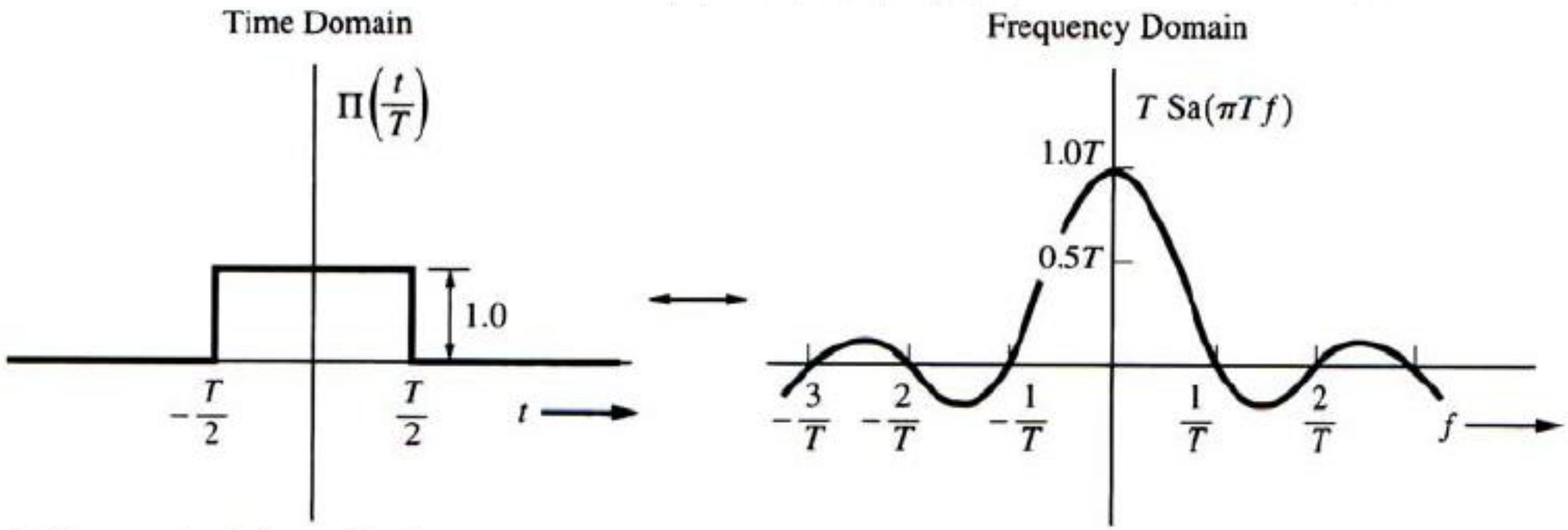
The spectrum is obtained by taking the Fourier transform of $w(t) = \Pi(t/T)$.

$$W(f) = \int_{-T/2}^{T/2} 1 e^{-j\omega t}\, dt = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$= T\,\frac{\sin(\omega T/2)}{\omega T/2} = T\,\mathrm{Sa}(\pi T f)$$

Thus,

$$\Pi\left(\frac{t}{T}\right) \qquad T\,\mathrm{Sa}(\pi T f)$$



(a) Rectangular Pulse and Its Spectrum

Note the inverse relationship between the pulse width T and the zero crossing 1/T

Example 2-5: Spectrum of a Rectangular Pulse

To find the spectrum of a Sa function we can use duality theorem From Table 2.1

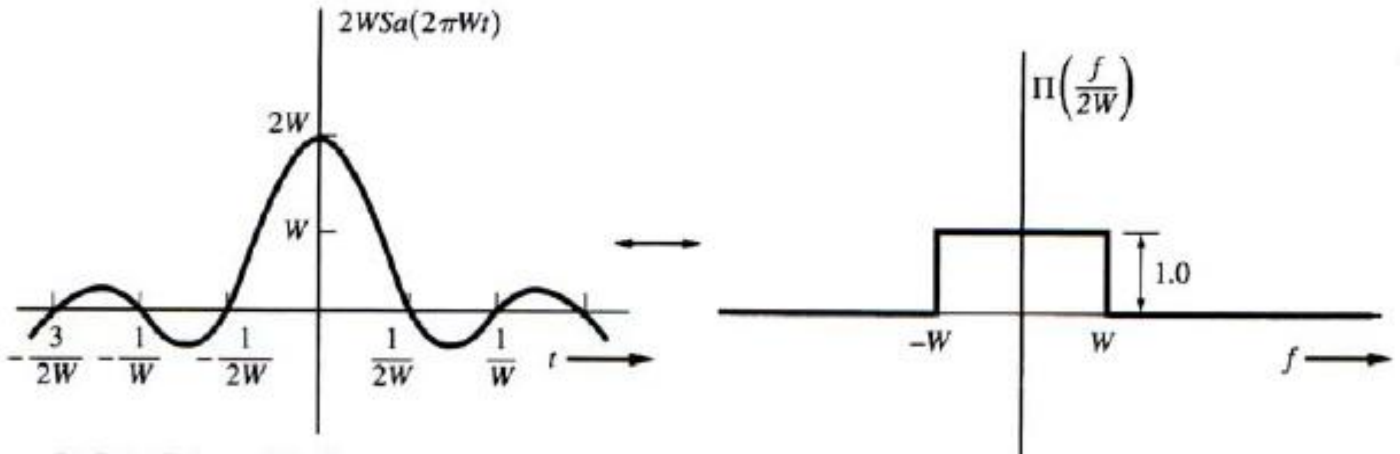
Duality: $W(t) \leftrightarrow w(-f)$

$$T\,Sa(\pi T f) \leftrightarrow \Pi\left(-\frac{f}{T}\right) = \Pi\left(\frac{f}{T}\right)$$ Because Π is an even and real function

Replacing the parameter T by $2W$, we obtain the Fourier transform pair.

$$2W Sa(2\pi W T) \leftrightarrow \Pi\left(\frac{f}{2W}\right) \quad (2\text{–}56)$$

where W is the absolute bandwidth in hertz. This Fourier transform pair is also shown in Fig. 2–6b.



(b) $Sa(x)$ Pulse and Its Spectrum

Example 2-4: Spectrum of a Rectangular Pulse

- The spectra shown in previous slides are real because the time domain pulse (rectangular pulse) is real and even
- If the pulse is offset in time domain to destroy the even symmetry, the spectra will be complex.
- Let us now apply the Time delay theorem of Table 2.1 to the Rectangular pulse

$$v(t) = \begin{cases} 1, & 0 < t < T \\ 0, & t \text{ elsewhere} \end{cases} = \Pi\left(\frac{t - T/2}{T}\right)$$

Time Delay Theorem: $w(t-T_d) \leftrightarrow W(f)\, e^{-j\omega T_d}$

When we apply this to:

$$T\,\mathrm{Sa}(\pi T t) \leftrightarrow \Pi\left(-\frac{f}{T}\right) = \Pi\left(\frac{f}{T}\right)$$

We get:

$$V(f) = Te^{-j\pi f T}\,\mathrm{Sa}(\pi T f)$$

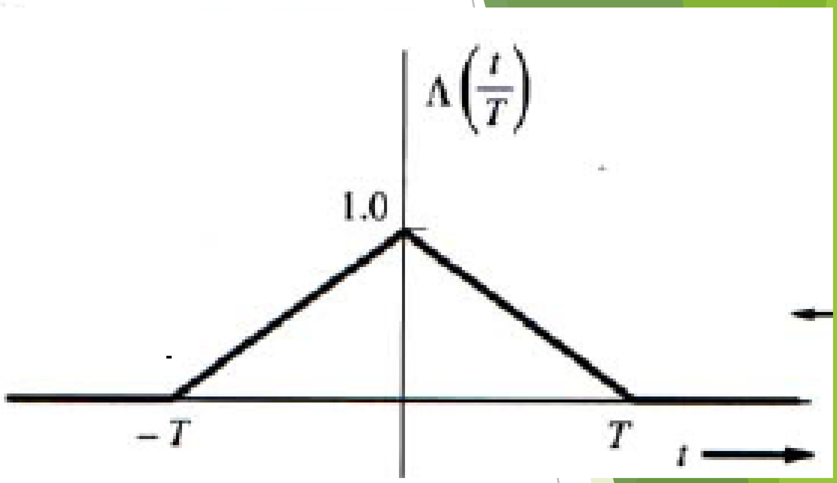
$$V(f) = \underbrace{[T\,\mathrm{Sa}(\pi f T)\cos(\pi f T)]}_{X(f)} + j\underbrace{[-T\,\mathrm{Sa}(\pi f T)\sin(\pi f T)]}_{Y(f)}$$

Example 2-6: Spectrum of a Triangular Pulse

$$w(t) = \Lambda(t/T)$$

$$\frac{dw(t)}{dt} = \frac{1}{T}u(t+T) - \frac{2}{T}u(t) + \frac{1}{T}u(t-T)$$

$$\frac{d^2w(t)}{dt^2} = \frac{1}{T}\delta(t+T) - \frac{2}{T}\delta(t) + \frac{1}{T}\delta(t-T)$$

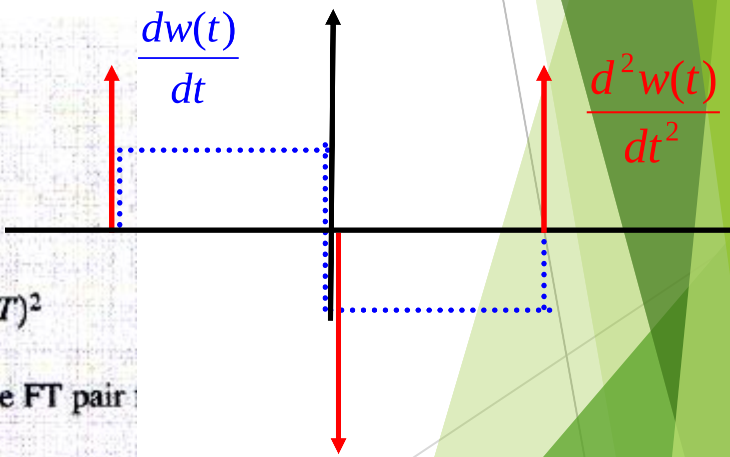


Using Table 2–2, we find that the FT pair for the second derivative is

$$\frac{d^2w(t)}{dt^2} \leftrightarrow \frac{1}{T}e^{j\omega T} - \frac{2}{T} + \frac{1}{T}e^{-j\omega T}$$

which can be rewritten as

$$\frac{d^2w(t)}{dt^2} \leftrightarrow \frac{1}{T}(e^{j\omega T/2} - e^{-j\omega T/2})^2 = \frac{-4}{T}(\sin \pi f T)^2$$



Referring to Table 2–1 and applying the integral theorem twice, we get the FT pair f
inal waveform:

$$w(t) \leftrightarrow \frac{-4}{T}\frac{(\sin \pi f T)^2}{(j2\pi f)^2}$$

Thus,

$$w(t) = \Lambda\left(\frac{t}{T}\right) \leftrightarrow T\,\mathrm{Sa}^2(\pi f T)$$

Spectrum of Rectangular, Sa and Triangular Pulses

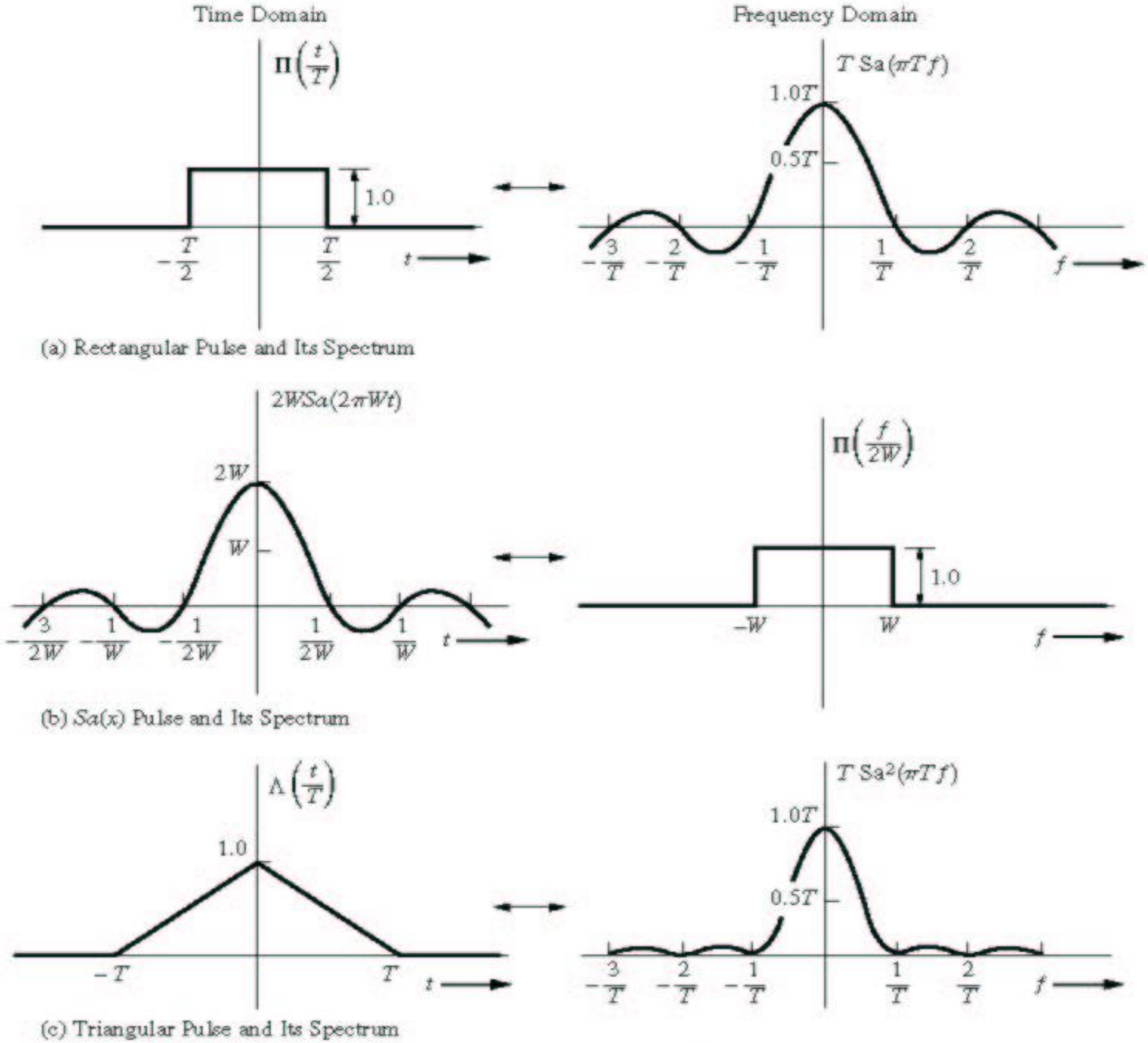


Figure 2–6 Spectra of rectangular, $(\sin x)/x$, and triangular pulses.

Table 2.2 Some FT pairs

Function	Time Waveform $w(t)$	Spectrum $W(f)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]^2$
Unit step	$u(t) \triangleq \begin{cases} +1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\mathrm{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sinc	$\mathrm{Sa}(2\pi W t)$	$\frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$
Phasor	$e^{j(\omega_0 t + \varphi)}$	$e^{j\varphi}\,\delta(f - f_0)$
Sinusoid	$\cos(\omega_c t + \varphi)$	$\frac{1}{2}e^{j\varphi}\,\delta(f - f_c) + \frac{1}{2}e^{-j\varphi}\,\delta(f + f_c)$
Gaussian	$e^{-\pi(t/t_0)^2}$	$t_0 e^{-\pi(f t_0)^2}$
Exponential, one-sided	$\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1 + j2\pi f T}$
Exponential, two-sided	$e^{-\lvert t \rvert/T}$	$\frac{2T}{1 + (2\pi f T)^2}$
Impulse train	$\sum_{k=-\infty}^{k=\infty} \delta(t - kT)$	$f_0 \sum_{n=-\infty}^{n=\infty} \delta(f - n f_0)$, where $f_0 = 1/T$