# SECTION B

# FOURIER TRANSFORM

# 2.1 FOURIER SERIES

- Usually, a signal is described as a function of time .
- There are some amazing advantages if a signal can be expressed in the frequency domain.
- Fourier transform analysis is named after Jean Baptiste Joseph Fourier (1768-1830).

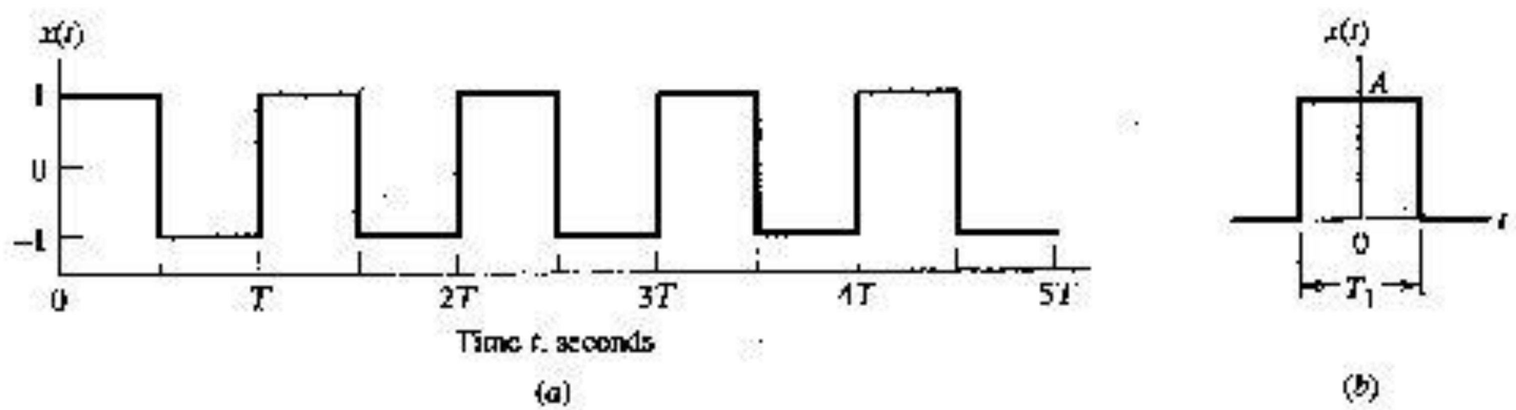
- A ***Fourier series*** (FS) is used for representing a continuous-time periodic signal as weighted superposition of sinusoids.
- **Periodic Signals** A continuous-time signal is said to be ***periodic*** if there exists a positive constant such that

$$x(t) = x(t + T_0)$$

where $T_0$ is the period of the signal.

- $T_0$ : fundamental Period
- $f_0 = \frac{1}{T_0}$ : fundamental frequency

- Example: Periodic and aperiodic signal



- $\{x_n\}$ are called the Fourier series coefficients of the signal *x(t)*.
- The quantity $f_0 = \frac{1}{T_0}$ is called the fundamental frequency of the signal *x(t)*
- The Fourier series expansion can be expressed in terms of angular frequency $\omega_0 = 2\pi f_0$ by

$$x_n = \frac{\omega_0}{2\pi} \int_{\alpha}^{\alpha + 2\pi/\omega_0} x(t) e^{-jn\omega_0 t} dt$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

- Discrete spectrum - We may write $x_n = |x_n| e^{j\angle x_n}$ , where $|x_n|$ gives the magnitude of the *n*th harmonic and $\angle x_n$ gives its phase.

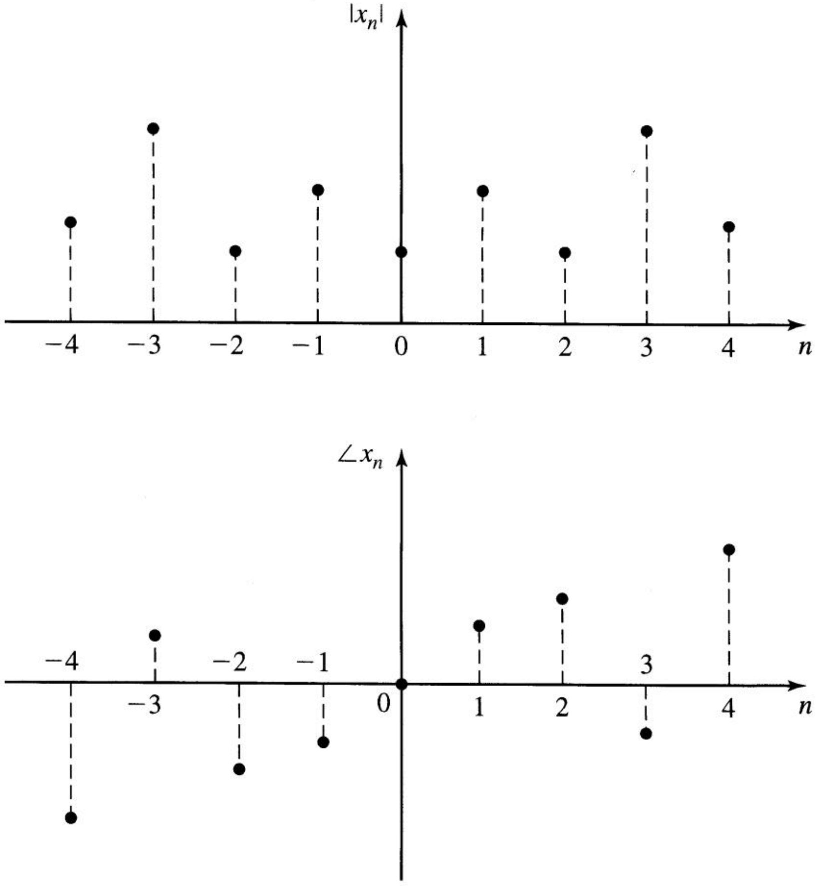



**Figure 2.1** The discrete spectrum of $x(t)$.

- Example: Let *x(t)* denote the periodic signal depicted in Figure 2.2

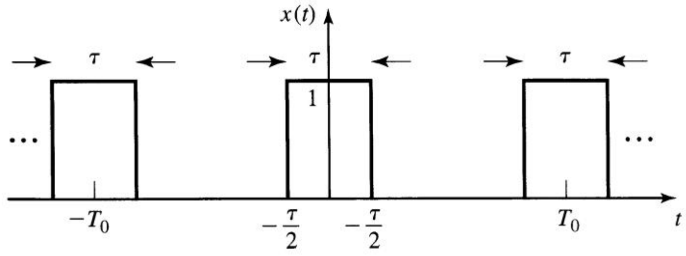



**Figure 2.2** Periodic signal $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_0}{\tau}\right), \quad T_0 > \tau,$$

where

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

is a rectangular pulse. Determine the Fourier series expansion for this signal.

Solution: We first observe that the period of the signal is $T_0$ and

$$x_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1 e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\frac{T_0}{-jn2\pi}\left[e^{-jn\frac{n\tau}{T_0}} - e^{jn\frac{n\tau}{T_0}}\right]$$

$$= \frac{1}{\pi n}\sin\left(\frac{n\pi\tau}{T_0}\right)$$

$$= \frac{\tau}{T_0}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Therefore, we have

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}$$

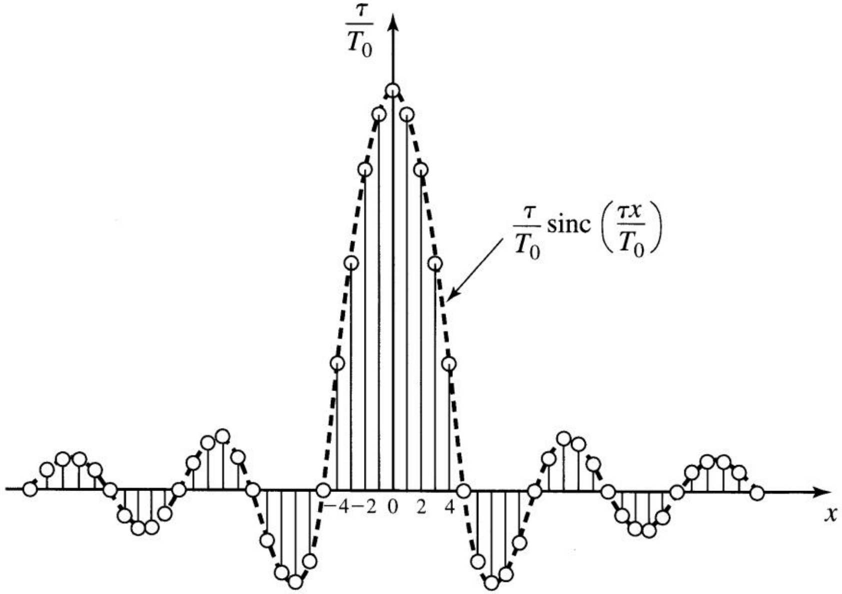



**Figure 2.3** The discrete spectrum of the rectangular pulse train.

Superposition of $x_M(t) = \sum_{n=-M}^{M} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}.$

$\tau = 0.5, T_0 = 2; \lim_{M\to\infty} x_M(t) = x(t).$

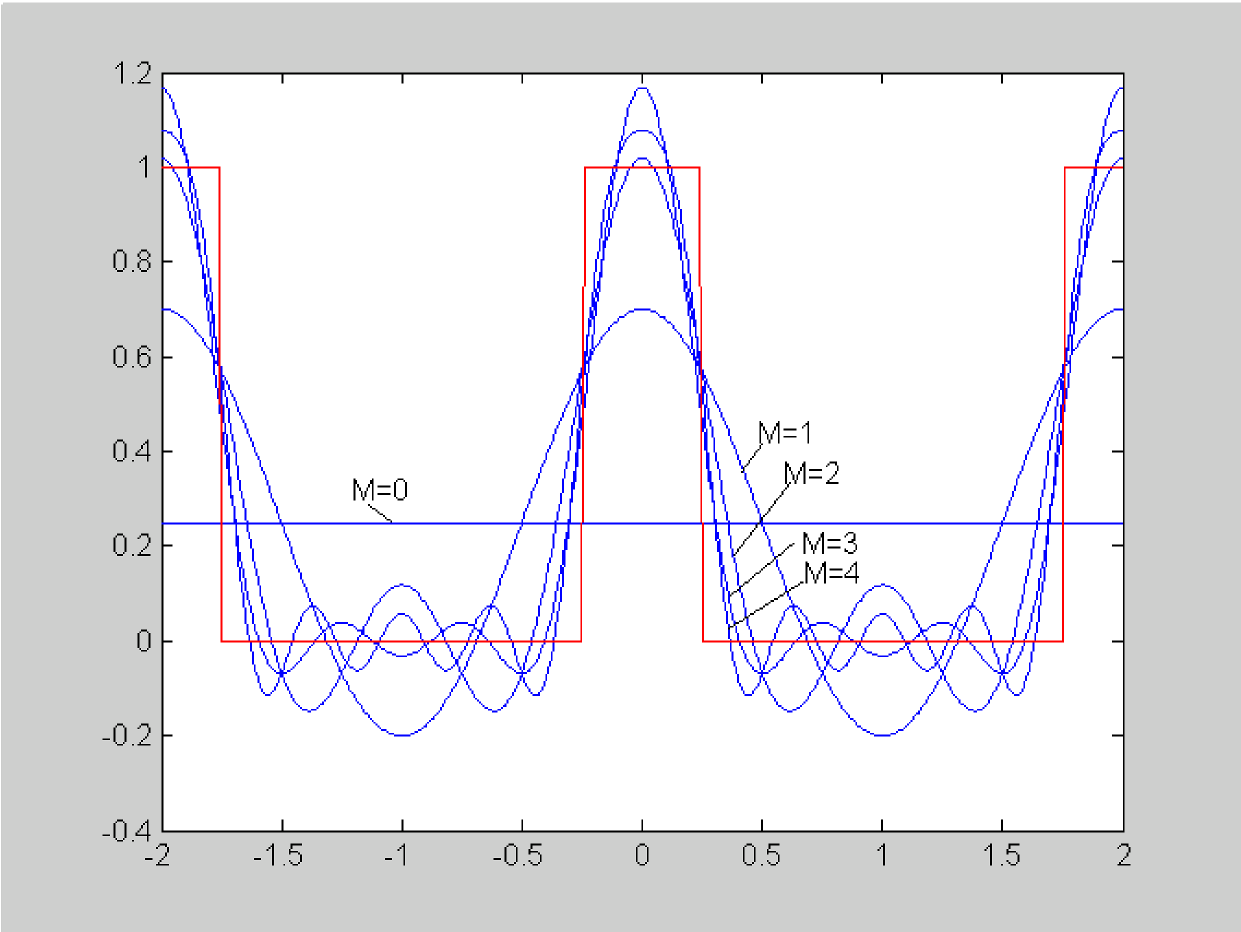


Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

| **Property** | **Periodic Signal** | **Fourier Series Coefficients** |
|---|---|---|
| | $\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$<br>$b_k$ |
| Linearity | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time-Shifting | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency-Shifting | $e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | $x(-t)$ | $a_{-k}$ |
| Time Scaling | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | $\int_T x(\tau)y(t-\tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | $\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$ | $\Re e\{a_k\}$<br>$j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

# 2.2.2 Basic Properties of the Fourier Transform

- **Linearity Property:** Given signals $x_1(t)$ and $x_2(t)$ with the Fourier transforms

$$\mathcal{F}[x_1(t)] = X_1(f)$$

$$\mathcal{F}[x_2(t)] = X_2(f).$$

The Fourier transform of $\alpha x_1(t) + \beta x_2(t)$ is

$$\mathcal{F}[\alpha x_1(t) + \beta x_2(t)] = \alpha X_1(f) + \beta X_2(f).$$

- **Duality Property:**

If $X(f) = \mathcal{F}[x(t)]$, then $x(f) = \mathcal{F}[X(-t)]$ and $x(-f) = \mathcal{F}[X(t)]$.

**Proof:**

$$\mathcal{F}[X(-t)] = \int_{-\infty}^{\infty} X(-t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} X(t) e^{j2\pi ft} dt$$
$$= x(f).$$

$$\mathcal{F}[X(t)] = \int_{-\infty}^{\infty} X(t) e^{-j2\pi ft} dt$$
$$= \int_{-\infty}^{\infty} X(t) e^{j2\pi(-f)t} dt$$
$$= x(-f).$$

- **Time Shift Property:** A shift of $t_0$ in the time origin causes a phase shift of $-2\pi f t_0$ in the frequency domain.

$$\mathcal{F}\,[x(t-t_0)] = e^{-j2\pi f t_0}\mathcal{F}\,[x(t)].$$

**Proof:**

$$\mathcal{F}\,[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f t}dt.$$

Let $t' = t - t_0$

$$\mathcal{F}\,[x(t-t_0)] = \int_{-\infty}^{\infty} x(t')e^{-j2\pi f (t'+t_0)}dt'$$

$$= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} x(t')e^{-j2\pi f t'}dt'$$

$$= e^{-j2\pi f t_0}\mathcal{F}\,[x(t)] = e^{-j2\pi f t_0}X(f).$$

- **Scaling Property:** For any real $a \neq 0$ , we have

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right).$$

- Proof:

Case 1: $a > 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$; we have $dt = (1/a)dt'$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(t') e^{-j2\pi(f/a)t'} dt' = \frac{1}{a} X\left(\frac{f}{a}\right).$$

Case 2: $a < 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$ ; we have $dt = (1/a)dt'$

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{\infty}^{-\infty} x(t') e^{-j2\pi(f/a)t'} dt' = -\frac{1}{a} X\left(\frac{f}{a}\right).$$

- **Convolution Property:** If the signal $x(t)$ and $y(t)$ both possess Fourier transforms, then

$$\mathcal{F}\,[x(t)*y(t)] = \mathcal{F}\,[x(t)]\;\mathcal{F}\,[y(t)] = X(f)\,Y(f).$$

Proof:

Convolution $$x(t)*y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

$$\mathcal{F}\,[x(t)*y(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\right)e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} y(t-\tau)e^{-j2\pi ft}dt\right)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(e^{-j2\pi f\tau}Y(f)\right)d\tau$$

$$= Y(f)\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}\,d\tau$$

$$= X(f)\,Y(f).$$

- **Modulation Property:** The Fourier transform of $x(t)\, e^{j2\pi f_0 t}$ is $X(f - f_0)$ , and the Fourier transform of

$$x(t)\cos(2\pi f_0 t) = x(t)\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

is

$$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0).$$

Proof:

$$\mathcal{F}\,[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f - f_0)t}\, dt$$

$$= X(f - f_0).$$

- **Parseval's Property:** If the Fourier transforms of $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df.$$

- proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(u)e^{j2\pi ut}du\right)\left(\int_{-\infty}^{\infty} Y(v)e^{j2\pi vt}dv\right)^* dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,e^{j2\pi ut}\;e^{-j2\pi vt}dt\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\left(\int_{-\infty}^{\infty} e^{j2\pi(u-v)t}dt\right)dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(v)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)\,Y^*(u)\,\delta(u-v)\,dv\,du$$

$$= \int_{-\infty}^{\infty} X(u)Y^*(u)du$$

$$= \int_{-\infty}^{\infty} X(f)Y^*(f)df.$$

- **Rayleigh's Property:** If *X(f)* is the Fourier transform of *x(t)*, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df.$$

Proof:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} X(f) X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 \, df.$$

**Parseval's Property**

- **Autocorrelation Property:** The (time) autocorrelation function of the aperiodic signal *x(t)* is denoted by $R_x(\tau)$ and is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt.$$

The autocorrelation property states that

$$\mathcal{F}\,[R_x(\tau)] = \left|X(f)\right|^2.$$

- **Differentiation Property:** The Fourier transform of the derivative of a signal can be obtained from the relation

$$\mathcal{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f\ X(f).$$

- **Integration Property:** The Fourier transform of the integral of a signal can be determined from the relation

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f).$$

- **Moments Property:** If $\mathcal{F}[x(t)] = X(f)$., then $\int_{-\infty}^{\infty} t^n x(t)dt$, the *n*th moment of *x(t)*, can be obtained from the relation

$$\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\bigg|_{f=0} .$$

**TABLE 2.1 TABLE OF FOURIER TRANSFORMS**

| Time Domain ($x(t)$) | Frequency Domain ($X(f)$) |
|---|---|
| $\delta(t)$ | $1$ |
| $1$ | $\delta(f)$ |
| $\delta(t-t_0)$ | $e^{-j2\pi f t_0}$ |
| $e^{j2\pi f_0 t}$ | $\delta(f-f_0)$ |
| $\cos(2\pi f_0 t)$ | $\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$ |
| $\sin(2\pi f_0 t)$ | $-\frac{1}{2j}\delta(f+f_0)+\frac{1}{2j}\delta(f-f_0)$ |
| $\Pi(t)=\begin{cases}1, & \|t\|<\frac{1}{2}\\ \frac{1}{2}, & t=\pm\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$ | $\text{sinc}(f)$ |
| $\text{sinc}(t)$ | $\Pi(f)$ |
| $\Lambda(t)=\begin{cases}t+1, & -1\le t<0\\ -t+1, & 0\le t<1\\ 0, & \text{otherwise}\end{cases}$ | $\text{sinc}^2(f)$ |
| $\text{sinc}^2(t)$ | $\Lambda(f)$ |
| $e^{-\alpha t}u_{-1}(t), \alpha>0$ | $\frac{1}{\alpha+j2\pi f}$ |
| $te^{-\alpha t}u_{-1}(t), \alpha>0$ | $\frac{1}{(\alpha+j2\pi f)^2}$ |
| $e^{-\alpha\|t\|}$ | $\frac{2\alpha}{\alpha^2+(2\pi f)^2}$ |
| $e^{-\pi t^2}$ | $e^{-\pi f^2}$ |
| $\text{sgn}(t)=\begin{cases}1, & t>0\\ -1, & t<0\\ 0, & t=0\end{cases}$ | $1/(j\pi f)$ |
| $u_{-1}(t)$ | $\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$ |
| $\delta'(t)$ | $j2\pi f$ |
| $\delta^{(n)}(t)$ | $(j2\pi f)^n$ |
| $\frac{1}{t}$ | $-j\pi\,\text{sgn}(f)$ |
| $\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_0)$ | $\frac{1}{T_0}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{T_0}\right)$ |

Table 4: **Basic Continuous-Time Fourier Transform Pairs**

| **Signal** | **Fourier transform** | **Fourier series coefficients (if periodic)** |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ | $a_1 = 1$<br>$a_k = 0, \quad \text{otherwise}$ |
| $\cos\omega_0 t$ | $\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0, \quad \text{otherwise}$ |
| $\sin\omega_0 t$ | $\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0, \quad \text{otherwise}$ |
| $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1, \quad a_k = 0, \ k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$<br>and<br>$x(t+T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi}\,\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t-nT)$ | $\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ | $\frac{2\sin\omega T_1}{\omega}$ | — |
| $\frac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t-t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t), \Re e\{a\} > 0$ | $\frac{1}{a+j\omega}$ | — |
| $te^{-at}u(t), \Re e\{a\} > 0$ | $\frac{1}{(a+j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$,<br>$\Re e\{a\} > 0$ | $\frac{1}{(a+j\omega)^n}$ | — |

Table 3: **Properties of the Continuous-Time Fourier Transform**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

| **Property** | **Aperiodic Signal** | **Fourier transform** |
|---|---|---|
| | $x(t)$<br>$y(t)$ | $X(j\omega)$<br>$Y(j\omega)$ |
| Linearity | $ax(t)+by(t)$ | $aX(j\omega)+bY(j\omega)$ |
| Time-shifting | $x(t-t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| Frequency-shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega-\omega_0))$ |
| Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| Time-Reversal | $x(-t)$ | $X(-j\omega)$ |
| Time- and Frequency-Scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$ |
| Convolution | $x(t)*y(t)$ | $X(j\omega)Y(j\omega)$ |
| Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}X(j\omega)*Y(j\omega)$ |
| Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| Integration | $\int_{-\infty}^{t}x(t)dt$ | $\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ |
| Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re e\{X(j\omega)\}=\Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\}=-\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$ |
| Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| Even-Odd Decomposition for Real Signals | $x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real]<br>$x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real] | $\Re e\{X(j\omega)\}$<br>$j\Im m\{X(j\omega)\}$ |

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$$