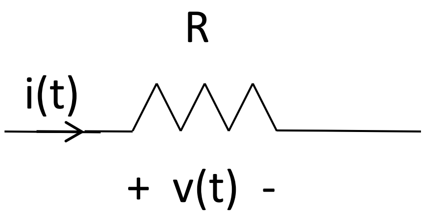
Stability

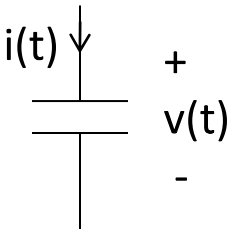
System External Stability

- Externally stable systems: **Bounded input** results in **bounded output** (system is said to be stable in the **BIBO** sense)

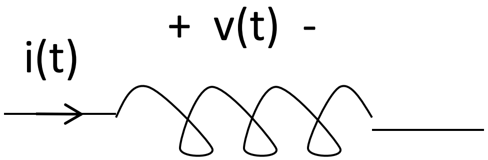
Electrical System



$$v(t) = R\,i(t)$$

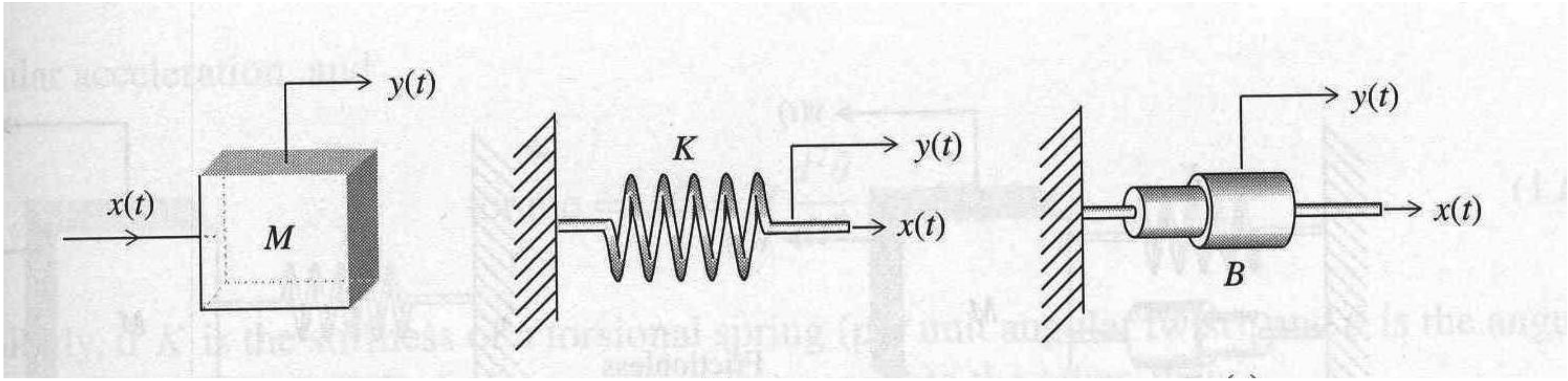


$$i(t) = C\frac{dv}{dt}$$



$$v(t) = L\frac{di}{dt}$$

Mechanical System



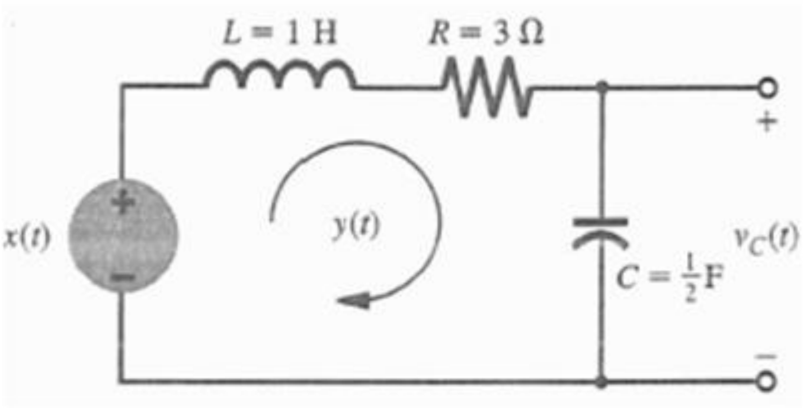
$$x(t) = M\,\ddot{y}(t) = M\,\frac{d^2y}{dt^2}$$

$$x(t) = k\;y(t)$$

$$x(t) = B\,\dot{y}(t) = B\,\frac{dy}{dt}$$

Linear Differential Systems (1)

- Many systems in electrical and mechanical engineering where input *x(t)* and output loop current *y(t)* are related by **differential equations**
- For example:



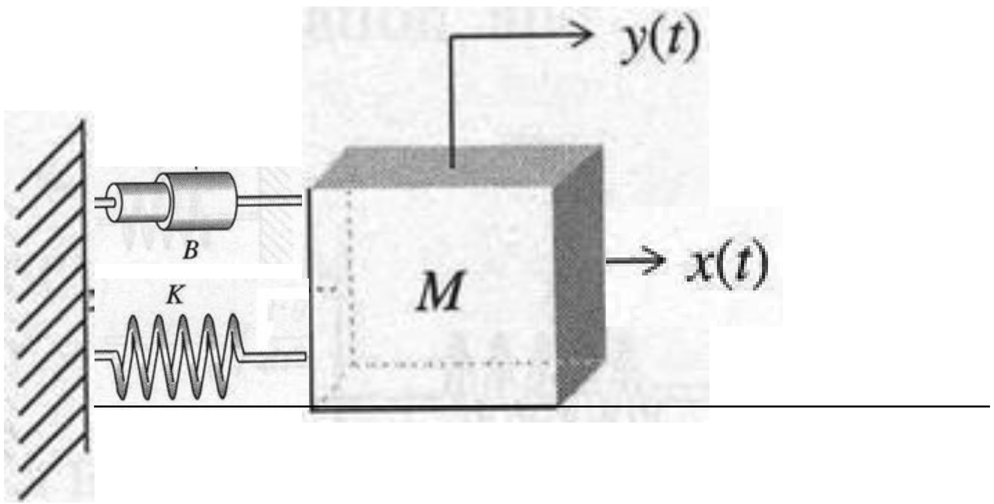
$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

$$\frac{dy}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt}$$

Linear Differential Systems (2)

Find the input-output relationship for the transational mechanical system shown below. The input is the force x(t), and the output is the mass position y(t)



Linear Differential Systems (3)

- In general, relationship between $x(t)$ and $y(t)$ in a **linear time-invariant** (**LTI**) differential system is given by (where all coefficients a_i and b_i are constants):

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t)$$

$$= b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

- Use compact notation **D** for **operator *d/dt***, i.e $\frac{dy}{dt} \equiv Dy(t)$ and $\frac{d^2 y}{dt^2} \equiv D^2 y(t)$ etc.
- We get: $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$

$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

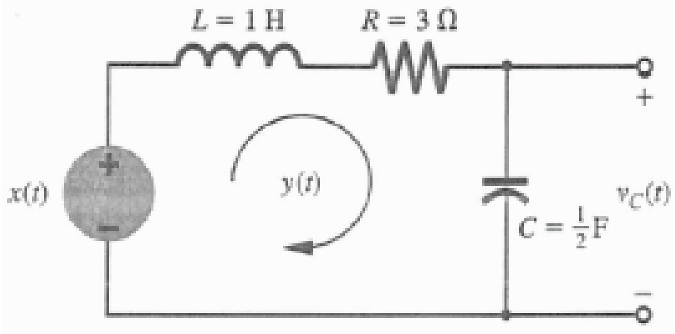
- or $$Q(D) y(t) = P(D) x(t)$$

$$Q(D) = D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N$$

$$P(D) = b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N$$

Linear Differential Systems (4)

- Let us consider this example again:



- The system equation is:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{dx}{dt}$$

- This can be re-written as:

$$\underbrace{(D^2 + 3D + 2)}_{Q(D)}y(t) = \underbrace{D}_{P(D)}x(t)$$

Also

$$\int_{-\infty}^{t} y(\tau)\,d\tau \equiv \frac{1}{D}y(t)$$

$$\frac{d}{dt}\left[\int_{-\infty}^{t} y(\tau)\,d\tau\right] = y(t)$$

- For this system, $N = 2, M = 1, a_1 = 3, a_2 = 2, b_1 = 1, b_2 = 0.$

- **For practical systems, M ≤ N**. It can be shown that if M > N, a LTI differential system acts as an (M – N)th-order **differentiator**.
- A differentiator is an unstable system because **bounded input** (e.g. a step input) results in an **unbounded output** (a Dirac impulse δ(t)).