

Properties of the Fourier Transform

Basic Properties of the Fourier Transform

- **Linearity Property:** Given signals $x_1(t)$ and $x_2(t)$ with the Fourier transforms

$$\mathcal{F} [x_1(t)] = X_1(f)$$

$$\mathcal{F} [x_2(t)] = X_2(f).$$

The Fourier transform of $\alpha x_1(t) + \beta x_2(t)$ is

$$\mathcal{F} [\alpha x_1(t) + \beta x_2(t)] = \alpha X_1(f) + \beta X_2(f).$$

- **Duality Property:**

If $X(f) = \mathcal{F} [x(t)]$, then $x(f) = \mathcal{F} [X(-t)]$ and $x(-f) = \mathcal{F} [X(t)]$.

Proof:

$$\begin{aligned}\mathcal{F} [X(-t)] &= \int_{-\infty}^{\infty} X(-t)e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} X(t)e^{j2\pi ft} dt \\ &= x(f).\end{aligned}$$

$$\begin{aligned}\mathcal{F} [X(t)] &= \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} X(t)e^{j2\pi(-f)t} dt \\ &= x(-f).\end{aligned}$$

- **Time Shift Property:** A shift of t_0 in the time origin causes a phase shift of $-2\pi ft_0$ in the frequency domain.

Proof:
$$\mathbf{F} [x(t - t_0)] = e^{-j2\pi ft_0} \mathbf{F} [x(t)].$$

$$\mathbf{F} [x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi ft} dt.$$

Let $t' = t - t_0$

$$\begin{aligned} \mathbf{F} [x(t - t_0)] &= \int_{-\infty}^{\infty} x(t') e^{-j2\pi f(t'+t_0)} dt' \\ &= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} x(t') e^{-j2\pi ft'} dt' \\ &= e^{-j2\pi ft_0} \mathbf{F} [x(t)] = e^{-j2\pi ft_0} X(f). \end{aligned}$$

- **Scaling Property:** For any real $a \neq 0$, we have

$$\mathcal{F} [x(at)] = \frac{1}{|a|} X \left(\frac{f}{a} \right).$$

- Proof:

Case 1:

Let $a > 0$; we have

$$\mathcal{F} [x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

$$t' = at \quad dt = (1/a) dt'$$

$$\mathcal{F} [x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(t') e^{-j2\pi(f/a)t'} dt' = \frac{1}{a} X \left(\frac{f}{a} \right).$$

Case 2:

$$a < 0$$

$$\mathcal{F} [x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt.$$

Let $t' = at$; we have

$$dt = (1/a) dt'$$

$$\mathcal{F} [x(at)] = \frac{1}{a} \int_{\infty}^{-\infty} x(t') e^{-j2\pi(f/a)t'} dt' = -\frac{1}{a} X \left(\frac{f}{a} \right).$$

- **Convolution Property:** If the signal $x(t)$ and $y(t)$ both possess Fourier transforms, then

Proof: $\mathcal{F} [x(t) * y(t)] = \mathcal{F} [x(t)] \mathcal{F} [y(t)] = X(f) Y(f).$

Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$$

$$\begin{aligned} \mathcal{F} [x(t) * y(t)] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau \right) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(t - \tau) e^{-j2\pi ft} dt \right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left(e^{-j2\pi f\tau} Y(f) \right) d\tau \\ &= Y(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \\ &= X(f) Y(f). \end{aligned}$$

- **Modulation Property:** The Fourier transform of $x(t) e^{j2\pi f_0 t}$ is $X(f - f_0)$ and the Fourier transform of

is
$$x(t) \cos(2\pi f_0 t) = x(t) \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

Proof:
$$\frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0).$$

$$\begin{aligned} \mathcal{F} [x(t) e^{j2\pi f_0 t}] &= \int_{-\infty}^{\infty} x(t) e^{j2\pi f_0 t} e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_0) t} dt \\ &= X(f - f_0). \end{aligned}$$

- **Parseval's Property:** If the Fourier transforms of $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df.$$

- proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t) y^*(t) dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(u) e^{j2\pi ut} du \right) \left(\int_{-\infty}^{\infty} Y(v) e^{j2\pi vt} dv \right)^* dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) Y^*(v) e^{j2\pi ut} e^{-j2\pi vt} dt dv du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) Y^*(v) \left(\int_{-\infty}^{\infty} e^{j2\pi(u-v)t} dt \right) dv du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) Y^*(v) \delta(u-v) dv du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) Y^*(u) \delta(u-v) dv du \\
&= \int_{-\infty}^{\infty} X(u) Y^*(u) du \\
&= \int_{-\infty}^{\infty} X(f) Y^*(f) df.
\end{aligned}$$

- **Rayleigh's Property:** If $X(f)$ is the Fourier transform of $x(t)$, then

Proof:
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} X(f) X^*(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Parseval's Property

- **Autocorrelation Property:** The (time) autocorrelation function of the **aperiodic** signal $x(t)$ is denoted by $R_x(\tau)$ and is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t - \tau) dt.$$

The autocorrelation property states that

$$\mathcal{F} [R_x(\tau)] = |X(f)|^2.$$

- **Differentiation Property:** The Fourier transform of the derivative of a signal can be obtained from the relation

$$\mathcal{F} \left[\frac{d}{dt} x(t) \right] = j2\pi f X(f).$$

- **Integration Property:** The Fourier transform of the integral of a signal can be determined from the relation

$$\mathbf{F} \left[\int_{-\infty}^t x(\tau) d\tau \right] = \frac{X(f)}{j2\pi f} + \frac{1}{2} X(0) \delta(f).$$

- **Moments Property:** If $\mathbf{F} [x(t)] = X(f)$, then $\int_{-\infty}^{\infty} t^n x(t) dt$, the n th moment of $x(t)$, can be obtained from the relation

$$\int_{-\infty}^{\infty} t^n x(t) dt = \left(\frac{j}{2\pi} \right)^n \frac{d^n}{df^n} X(f) \Big|_{f=0}.$$

TABLE 2.1 TABLE OF FOURIER TRANSFORMS

Time Domain ($x(t)$)	Frequency Domain ($X(f)$)
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$-\frac{1}{2j}\delta(f + f_0) + \frac{1}{2j}\delta(f - f_0)$
$\Pi(t) = \begin{cases} 1, & t < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$	$\text{sinc}(f)$
$\text{sinc}(t)$	$\Pi(f)$
$\Lambda(t) = \begin{cases} t + 1, & -1 \leq t < 0 \\ -t + 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$	$\text{sinc}^2(f)$
$\text{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t} u_{-1}(t), \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$
$t e^{-\alpha t} u_{-1}(t), \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha t }$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0 \end{cases}$	$1/(j\pi f)$
$u_{-1}(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\frac{1}{t}$	$-j\pi \text{sgn}(f)$
$\sum_{n=-\infty}^{n=+\infty} \delta(t - nT_0)$	$\frac{1}{T_0} \sum_{n=-\infty}^{n=+\infty} \delta(f - \frac{n}{T_0})$

Table 4: Basic Continuous-Time Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi \delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & t < T_1 \\ 0, & T_1 < t \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \omega < W \\ 0, & \omega > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at} u(t), \Re\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at} u(t), \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$ ECE, DCE	—

Table 3: Properties of the Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Property	Aperiodic Signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time-shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^t x(t)dt$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega} X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re\{X(j\omega)\} = \Re\{X(-j\omega)\} \\ \Im\{X(j\omega)\} = -\Im\{X(-j\omega)\} \\ X(j\omega) = X(-j\omega) \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [x(t) real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [x(t) real]	$\Re\{X(j\omega)\}$ $j\Im\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transform for Periodic Signals

- Let $x(t)$ be a periodic signal with period T_0 . Let $\{x_n\}$ denote the Fourier series coefficients corresponding to this signal. Then

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi\frac{n}{T_0}t}.$$

- Since

$$\mathcal{F} \left[e^{j2\pi\frac{n}{T_0}t} \right] = \delta \left(f - \frac{n}{T_0} \right),$$

we obtain

$$X(f) = \sum_{n=-\infty}^{\infty} x_n \delta \left(f - \frac{n}{T_0} \right).$$

- If we define the truncated signal $x_{T_0}(t)$ as

$$x_{T_0}(t) = \begin{cases} x(t), & -\frac{T_0}{2} < t \leq \frac{T_0}{2} \\ 0, & \text{otherwise.} \end{cases}$$

we may have

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x_{T_0}(t - nT_0) \\ &= x_{T_0}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0). \end{aligned}$$

- By using the convolution theorem, we obtain

$$\begin{aligned} X(f) &= X_{T_0}(f) \left[\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta \left(f - \frac{n}{T_0} \right) \right] \\ &= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X_{T_0}(f) \delta \left(f - \frac{n}{T_0} \right) \\ &= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X_{T_0} \left(\frac{n}{T_0} \right) \delta \left(f - \frac{n}{T_0} \right). \end{aligned}$$

Comparing this result with $X(f) = \sum_{n=-\infty}^{\infty} x_n \delta \left(f - \frac{n}{T_0} \right)$,

we conclude

$$x_n = \frac{1}{T_0} X_{T_0} \left(\frac{n}{T_0} \right).$$

- Alternative way to find the Fourier series coefficients. Given the periodic signal $x(t)$, we carry out the following steps to find a_n :

1. Find the truncated x^n signal $x_{T_0}(t)$.

2. Determine the Fourier transform $X_{T_0}(f)$ of the truncated signal.

3. Evaluate the Fourier transform of the truncated signal $X_{T_0}(f)$ at $f = \frac{n}{T_0}$, to obtain the n th harmonic and multiply by $\frac{1}{T_0}$.

- Example 2.2.3: Determine the Fourier series coefficients of the signal $x(t)$ shown in Figure 2.2.

Solution: The truncated signal is

$$x_{T_0}(t) = \Pi\left(\frac{t}{\tau}\right)$$

and its Fourier transform is

Therefore,

$$X_{T_0}(f) = \tau \operatorname{sinc}(\tau f).$$

$$x_n = \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right).$$

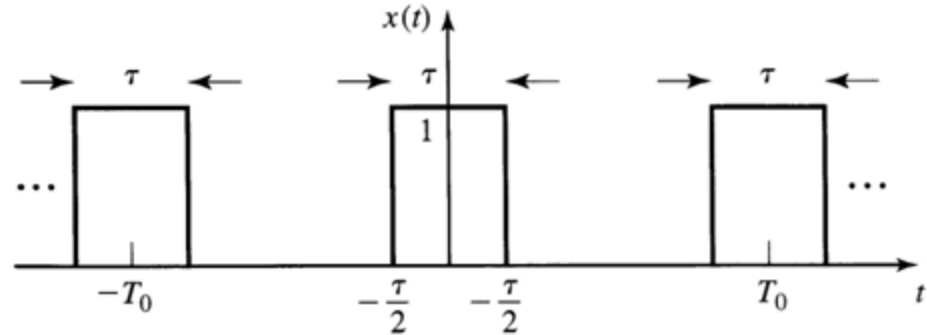


Figure 2.2 Periodic signal $x(t)$.