

FOURIER TRANSFORMS

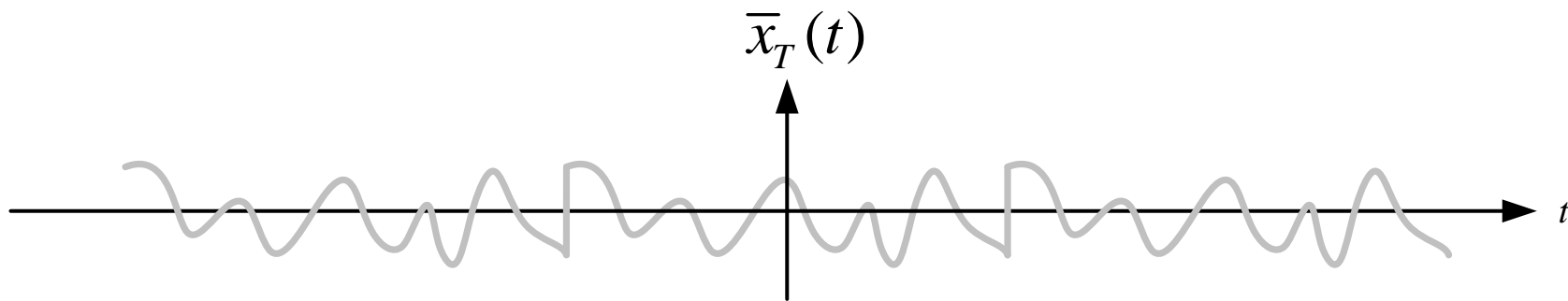
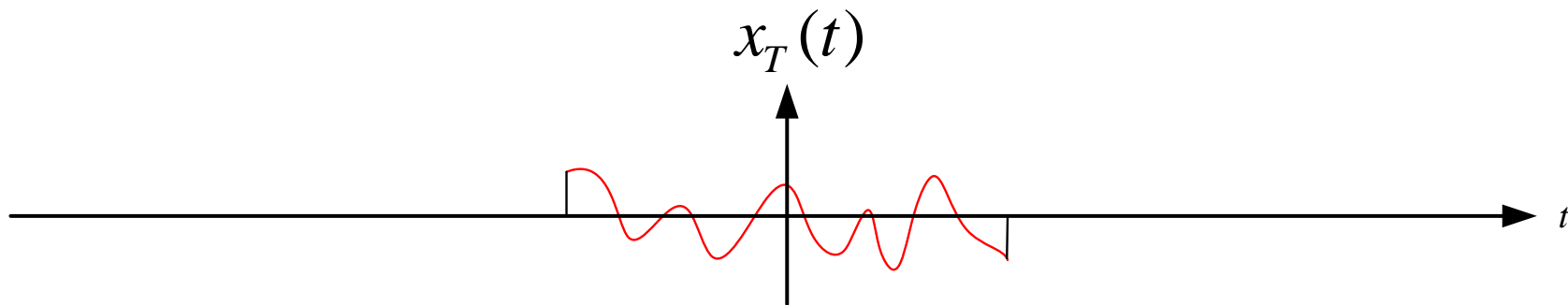
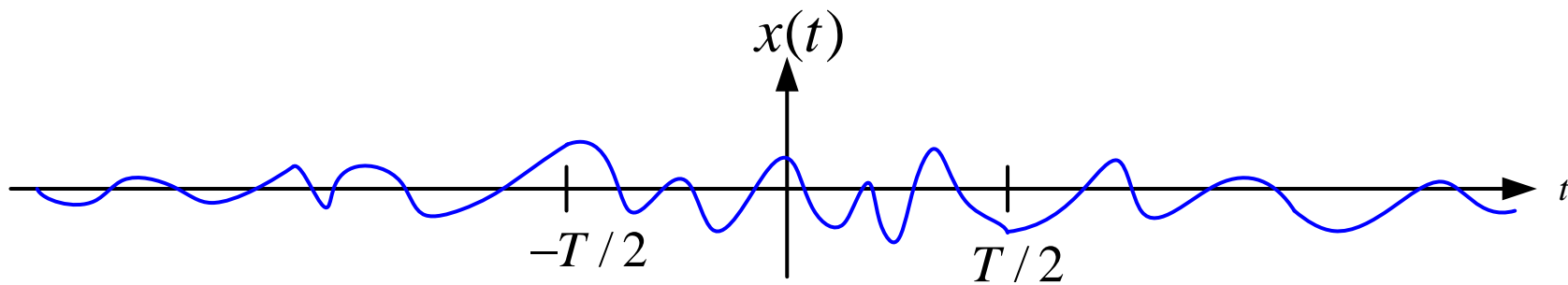
FOURIER TRANSFORMS

- Fourier transform is the extension of Fourier series to **periodic** and **aperiodic** signals.
- The signals are expressed in terms of complex exponentials of various frequencies, but these frequencies are not discrete.
- The extension of the Fourier series to aperiodic signals can be done by extending the period to infinity.
- The signal has a continuous spectrum as opposed to a discrete spectrum.

- Assume that the Fourier series of periodic extension of the nonperiodic signal $x(t)$ exists.
- Define $x_T(t)$ as the truncation of $x(t)$ over

$$-\frac{T}{2} < t < \frac{T}{2} \quad , \text{ i.e.,}$$

$$x_T(t) = \Pi\left(\frac{t}{T}\right)x(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$



- Denote the periodic signal

$$\bar{x}_T(t) = \sum_{k=-\infty}^{\infty} x_T(t - kT).$$

- Conversely, we may express the truncated signal by

$$x_T(t) = \begin{cases} \bar{x}_T(t), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases}$$

- If we let the period T approach infinity, then in the limit, the periodic signal approximately becomes the aperiodic signal

$$x(t) = \lim_{T \rightarrow \infty} x_T(t) = \lim_{T \rightarrow \infty} \bar{x}_T(t).$$

- This periodic signal with fundamental period T has a complex exponential Fourier series that is given by

$$\bar{x}_T(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t},$$

$$x_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \bar{x}_T(t) e^{-j2\pi n f_0 t} dt.$$

- As far as the integration is concerned, the integrand on this integral can be rewritten as

$$x_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \bar{x}_T(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-j2\pi n f_0 t} dt.$$

- Define

$$X_T(f) = \int_{-\infty}^{\infty} x_T(t) e^{-j2\pi f t} dt.$$

- We have

$$x_n = \frac{1}{T} X_T(nf_0).$$

- We have the Fourier series representation

$$\bar{x}_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X_T(nf_0) e^{j2\pi nf_0 t} = \sum_{n=-\infty}^{\infty} X_T(nf_0) e^{j2\pi nf_0 t} f_0.$$

- Taking the limit, we obtain

$$x(t) = \lim_{T \rightarrow \infty} \bar{x}_T(t) = \lim_{T \rightarrow \infty} \sum_{n=-\infty}^{\infty} X_T(nf_0) e^{j2\pi nf_0 t} f_0$$

- As $T \rightarrow \infty$, $f_0 \rightarrow 0$. That is, in the limit, the frequency spacing becomes small.

- The summation turns to become an integral

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

- $x(t)$ is the inverse Fourier transform of $X(f)$.
- The Fourier transform of $x(t)$ is

$$X(f) = \lim_{T \rightarrow \infty} X_T(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

- **Definition III.** Suppose that, $x(t)$, $-\infty < t < \infty$ is a signal such that it is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty.$$

Then the *Fourier transform* of $x(t)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

The inverse Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

Fourier transform - Sufficient conditions

➤ The waveform $w(t)$ is Fourier transformable if it satisfies both **Dirichlet conditions**:

- 1) Over any time interval of finite length, the function $w(t)$ is single valued with a finite number of maxima and minima, and the number of discontinuities (if any) is finite.
- 2) $w(t)$ is absolutely integrable. That is,

$$\int_{-\infty}^{\infty} |w(t)| dt < \infty$$

- Above conditions are **sufficient**, but **not necessary**.
- A weaker sufficient condition for the existence of the Fourier transform is:

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt < \infty$$

Finite Energy

- where E is the normalized energy.
- This is the finite-energy condition that is satisfied by all physically realizable waveforms.
- **Conclusion:** All physical waveforms encountered in engineering practice are Fourier transformable.

- Observations

- $X(f)$ is in general a complex function. The function $X(f)$ is sometimes referred to as the *spectrum* of the signal $x(t)$.
- To denote that $X(f)$ is the Fourier transform of $x(t)$, the following notation is frequently employed

$$X(f) = \mathbf{F} [x(t)].$$

- To denote that $x(t)$ is the inverse Fourier transform of $X(f)$, the following notation is used

$$x(t) = \mathbf{F}^{-1}[X(f)].$$

- Sometimes the following notation is used as a shorthand for both relations

$$x(t) \Leftrightarrow X(f).$$

- The Fourier transform and the inverse Fourier transform relations can be written as

$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \right] e^{j2\pi ft} df \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df \right] x(\tau) d\tau.\end{aligned}$$

On the other hand,

$$x(t) = \int_{-\infty}^{\infty} \delta(t - \tau) x(\tau) d\tau,$$

where $\delta(t)$ is the unit impulse. From above equation, we may have

$$\delta(t - \tau) = \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df,$$

or, in general

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df.$$

Hence, the spectrum of $\delta(t)$ is equal to unity over all frequencies.

Example 2.2.1: Determine the Fourier transform of the signal $\Pi(t)$.

Solution: We have

$$\begin{aligned} \mathcal{F} [\Pi(t)] &= \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(t) e^{-j2\pi ft} dt \\ &= \frac{1}{-j2\pi f} \left[e^{-j\pi f} - e^{j\pi f} \right] \\ &= \frac{\sin(\pi f)}{\pi f} \\ &= \text{sinc}(f). \end{aligned}$$

- The Fourier transform of $\Pi(t)$.

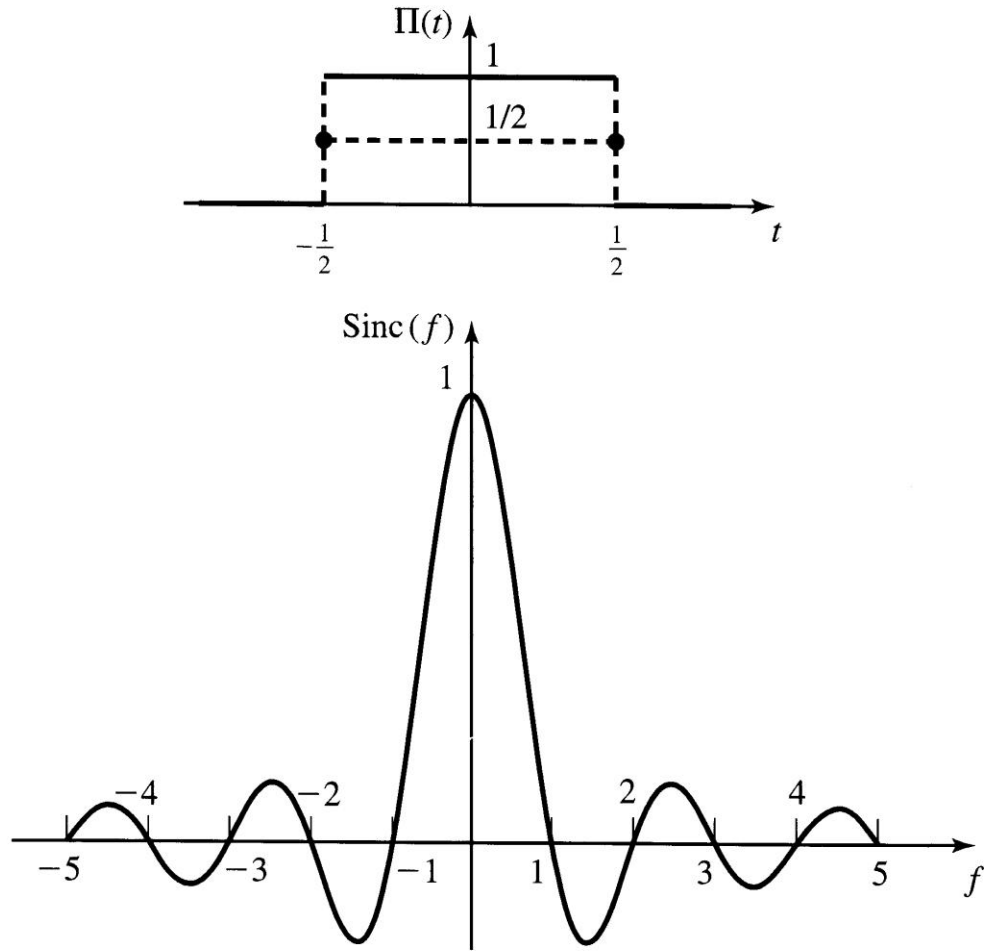


Figure 2.6 $\Pi(t)$ and its Fourier transform.

Example 2.2.2: Find the Fourier transform of the impulse signal $x(t) = \delta(t)$.

Solution: The Fourier transform can be obtained by

$$\begin{aligned} \mathcal{F} [\delta(t)] &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \\ &= 1. \end{aligned}$$

Similarly, from the relation

$$\int_{-\infty}^{\infty} \delta(f) e^{j2\pi ft} df = 1.$$

We conclude that

$$\mathcal{F} [1] = \delta(f).$$

- The Fourier transform of $\delta(t)$.

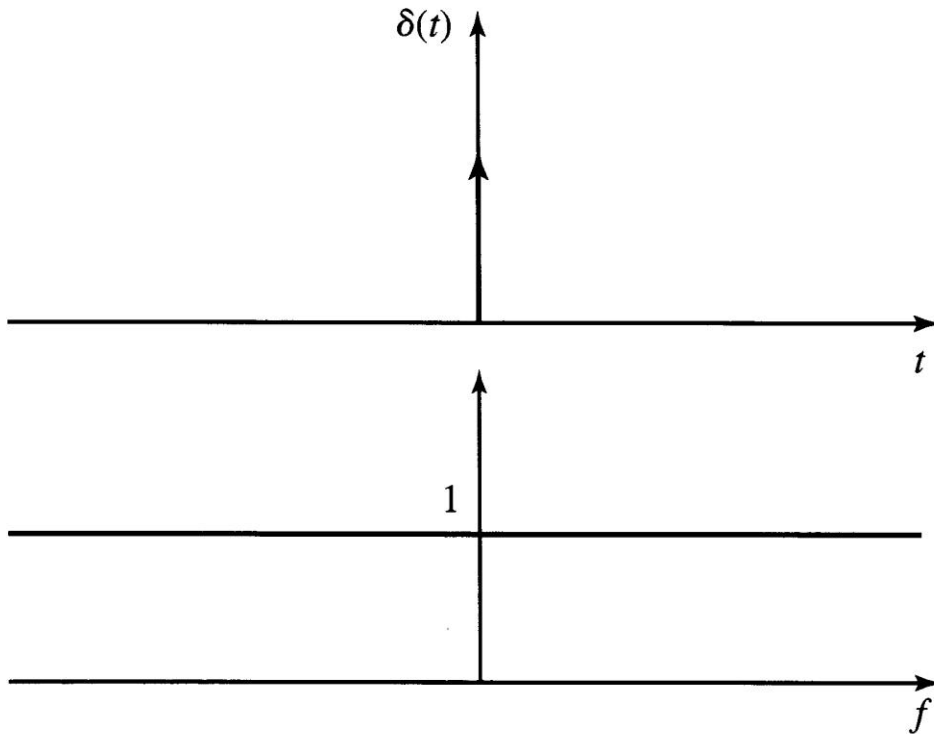


Figure 2.7 Impulse signal and its spectrum.