Fourier Analysis

FOURIER SERIES

- Usually, a signal is described as a function of time .
- There are some amazing advantages if a signal can be expressed in the frequency domain.
- Fourier transform analysis is named after Jean Baptiste Joseph Fourier (1768-1830).

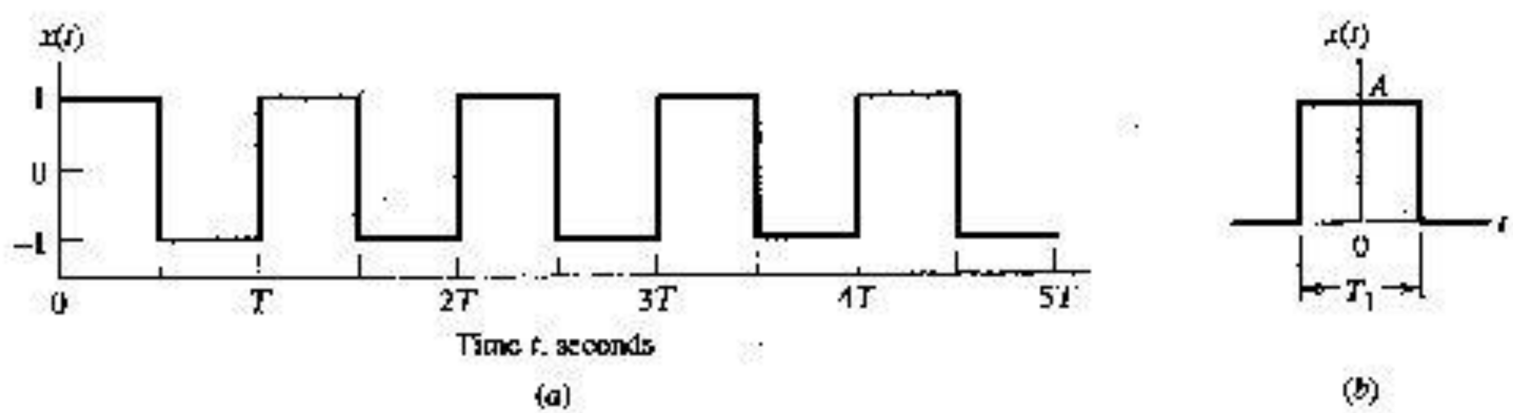
- A ***Fourier series*** (FS) is used for representing a continuous-time periodic signal as weighted superposition of sinusoids.
- **Periodic Signals** A continuous-time signal is said to be ***periodic*** if there exists a positive constant such that

$$x(t) = x(t + T_0)$$

where T_0 is the period of the signal.

- T_0 : fundamental Period
- $f_0 = \frac{1}{T_0}$: fundamental frequency

- Example: Periodic and aperiodic signal

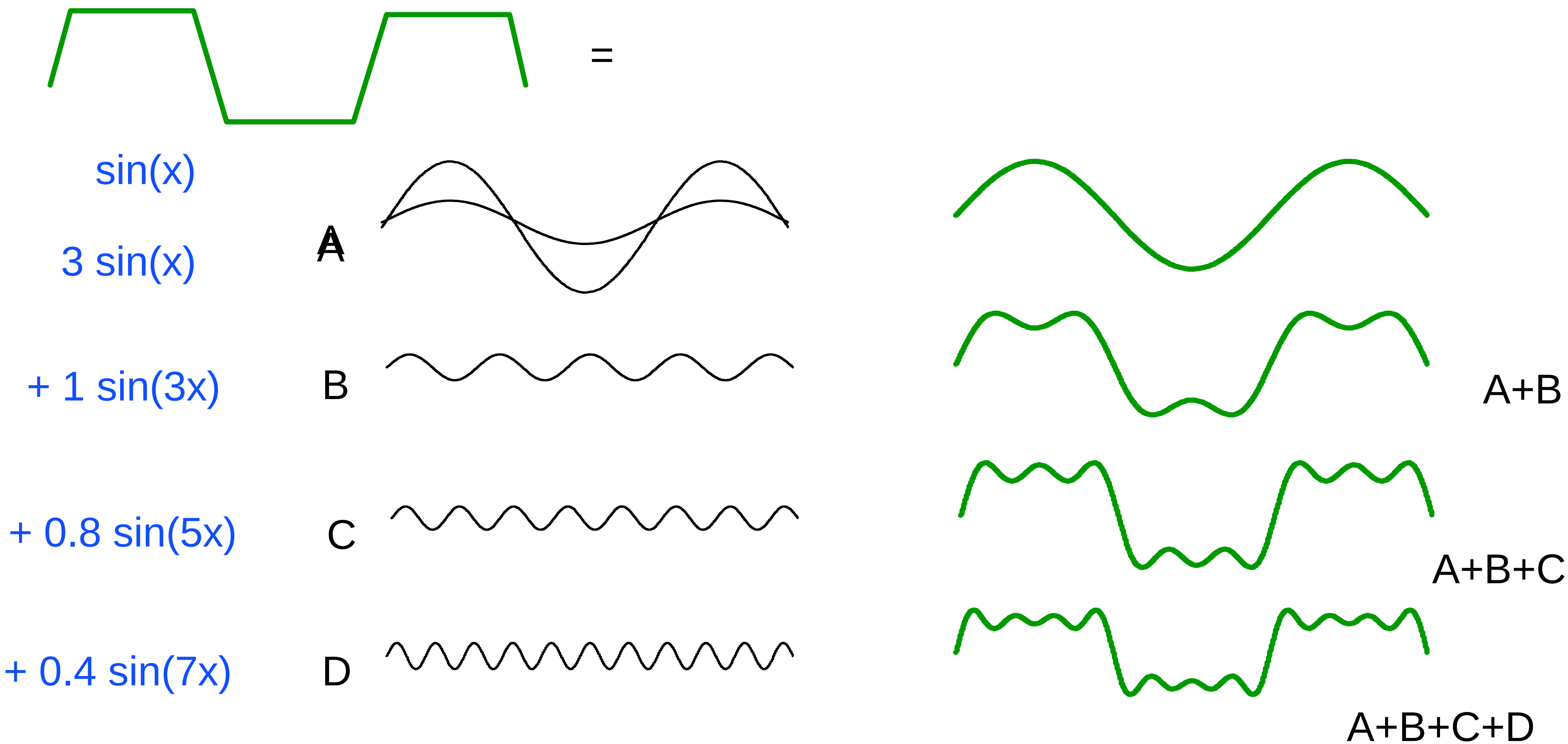


After the analysis, we obtain the following information about the signal:

I. What all frequency components are presenting the signal?
II. Their amplitude and
III. The relative phase difference between these frequency components.

All the frequency components are nothing else but sine waves at those frequencies.

A sum of Sines and Cosines



Existence of the Fourier Series

- Existence $\int_0^{T_0} |f(t)|\, dt < \infty$
- Convergence for all *t* $|f(t)| < \infty \;\; \forall t$
- Finite number of maxima and minima in one period of *f*(*t*)
- These are known as the Dirichlet conditions

Fourier Series

- General representation of a periodic signal

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$

- Fourier series coefficients

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\omega_0 t)dt$$

- Polar Form of Fourier series

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

where $c_0 = a_0, c_n = \sqrt{a_n^2 + b_n^2}$, and

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

- *$\{x_n\}$* are called the Fourier series coefficients of the signal *$x(t)$.*
- The quantity $f_0 = \frac{1}{T_0}$ is called the fundamental frequency of the signal *$x(t)$*
- The Fourier series expansion can be expressed in terms of angular frequency $\omega_0 = 2\pi f_0$ by

$$x_n = \frac{\omega_0}{2\pi} \int_{\alpha}^{\alpha + 2\pi/\omega_0} x(t) e^{-jn\omega_0 t} dt$$

and

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

- Discrete spectrum - We may write $x_n = |x_n| e^{j\angle x_n}$, where $|x_n|$ gives the magnitude of the *n*th harmonic and $\angle x_n$ gives its phase.

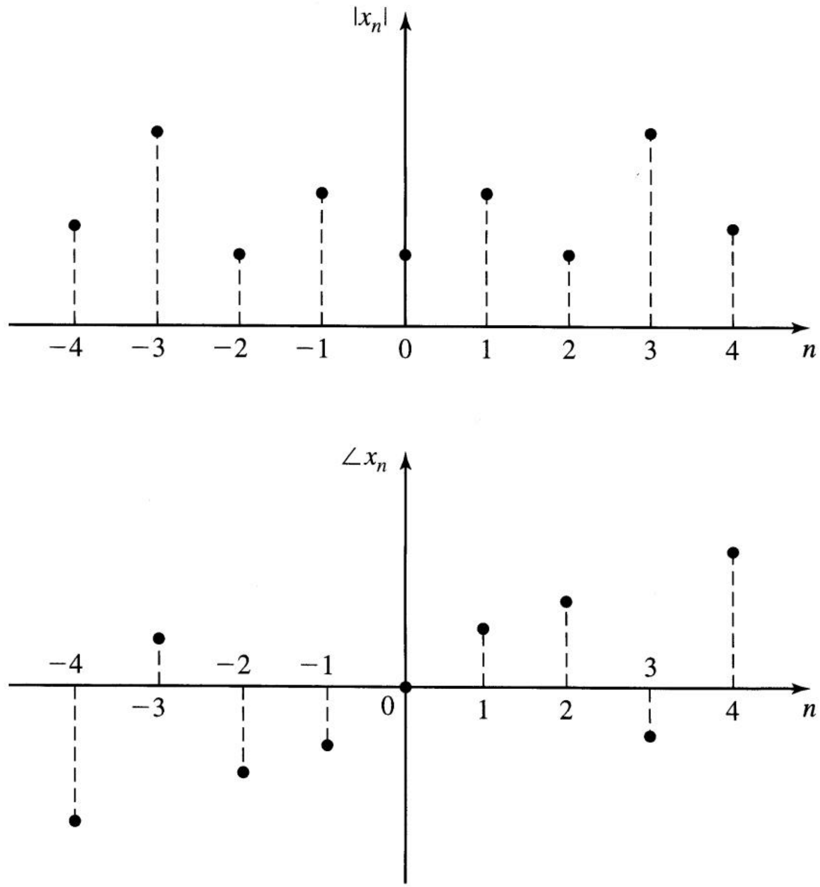


Figure 2.1 The discrete spectrum of $x(t)$.

- Example: Let *x(t)* denote the periodic signal depicted in Figure 2.2

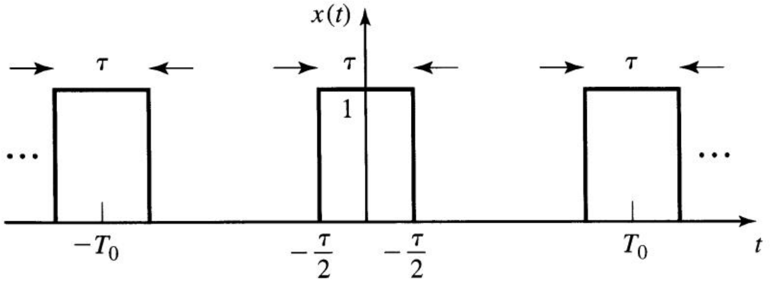


Figure 2.2 Periodic signal $x(t)$.

$$x(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_0}{\tau}\right), \quad T_0 > \tau,$$

where

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

is a rectangular pulse. Determine the Fourier series expansion for this signal.

Solution: We first observe that the period of the signal is T_0 and

$$x_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1 e^{-jn\frac{2\pi t}{T_0}} dt$$

$$= \frac{1}{T_0}\frac{T_0}{-jn2\pi}\left[e^{-jn\frac{n\tau}{T_0}} - e^{jn\frac{n\tau}{T_0}}\right]$$

$$= \frac{1}{\pi n}\sin\left(\frac{n\pi\tau}{T_0}\right)$$

$$= \frac{\tau}{T_0}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Therefore, we have

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{jn\frac{2\pi t}{T_0}}$$

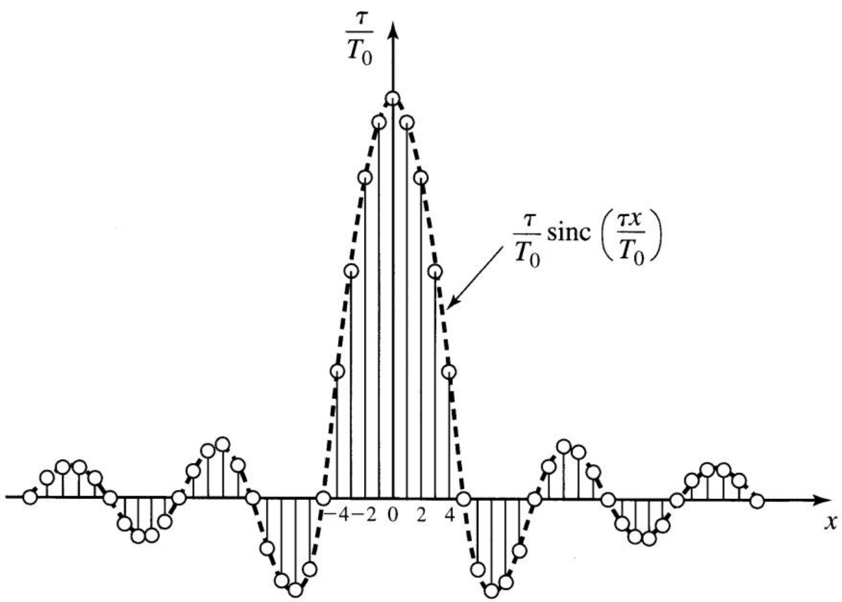
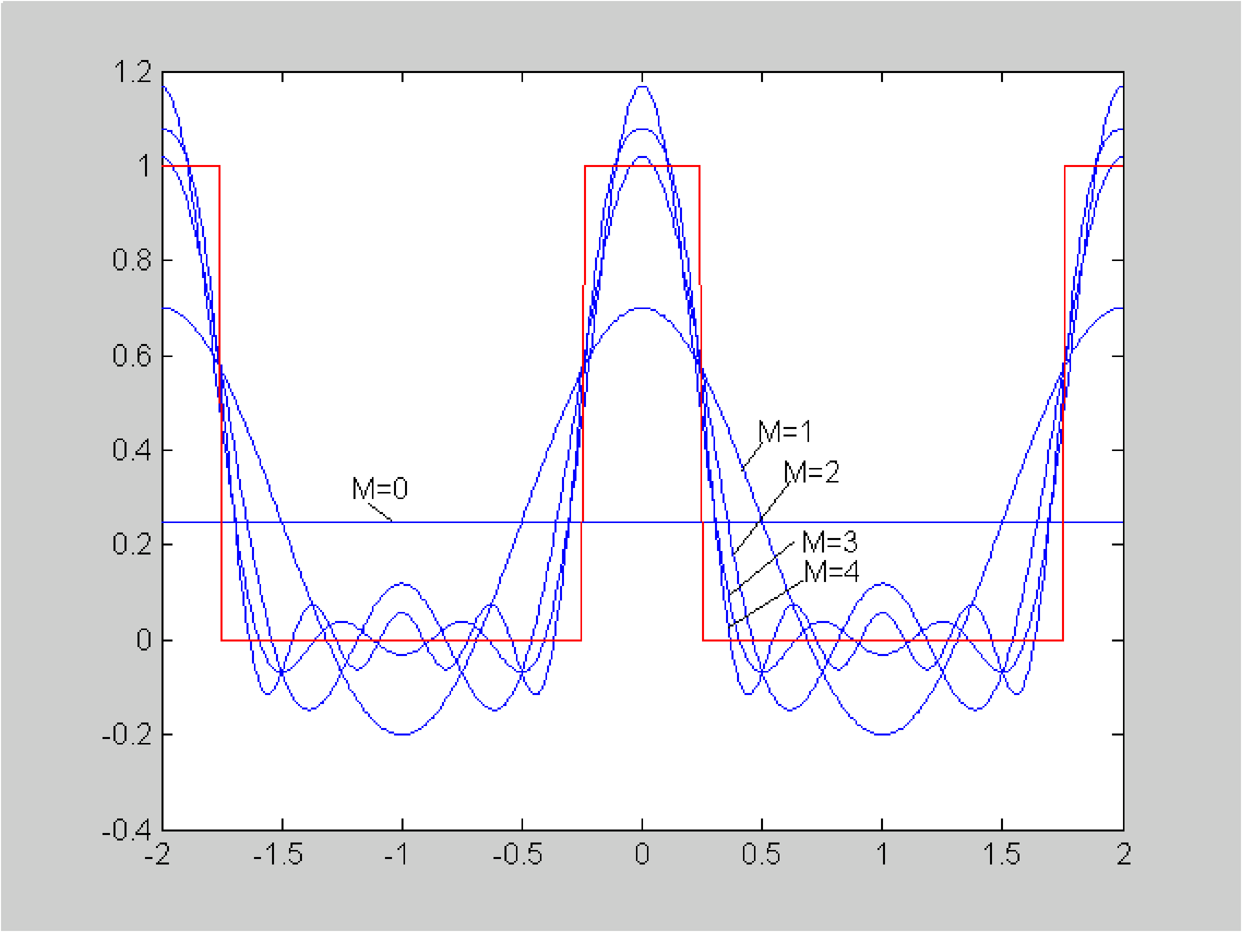
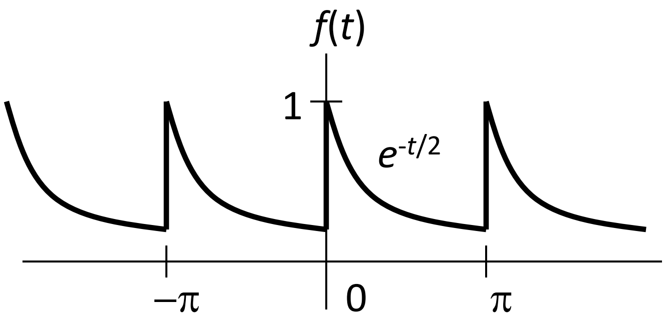


Figure 2.3 The discrete spectrum of the rectangular pulse train.



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
M=0
M=1
M=2
M=3
M=4

Example #1



- Fundamental period

  $T_0 = \pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/\pi$ Hz

  $\omega_0 = 2\pi/T_0 = 2$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} dt = -\frac{2}{\pi}\left(e^{-\frac{\pi}{2}} - 1\right) \approx 0.504$$

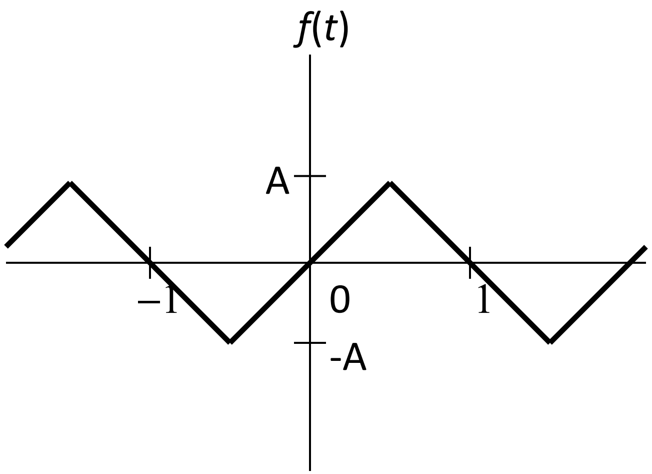
$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} \cos(2nt)\, dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} \sin(2nt)\, dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

a_n and b_n decrease in amplitude as $n \to \infty$.

$$f(t) = 0.504\left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right]$$

Example #2



- Fundamental period

  $T_0 = 2$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/2$ Hz

  $\omega_0 = 2\pi/T_0 = \pi$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n t) + b_n \sin(\pi n t)$$

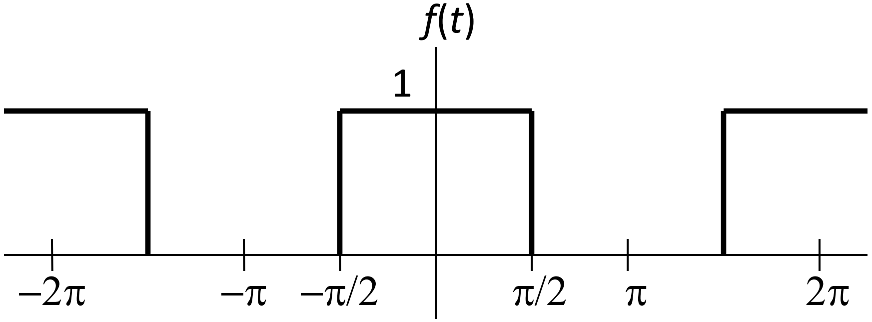
$$a_0 = 0 \quad \text{(by inspection of the plot)}$$

$$a_n = 0 \quad \text{(because it is odd symmetric)}$$

$$b_n = \frac{2}{\pi} \int_{-1/2}^{1/2} 2At \sin(\pi n t)\, dt + \frac{2}{\pi} \int_{1/2}^{3/2} (2A - 2At) \sin(\pi n t)\, dt$$

$$b_n = \begin{cases} 0 & n \text{ is even} \\ \dfrac{8A}{n^2\pi^2} & n = 1,5,9,13,\ldots \\ -\dfrac{8A}{n^2\pi^2} & n = 3,7,11,15,\ldots \end{cases}$$

Example #3



- Fundamental period

  $T_0 = 2\pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/2\pi$ Hz

  $\omega_0 = 2\pi/T_0 = 1$ rad/s

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3,7,11,15,\ldots \\ -\pi & n = 3,7,11,15,\ldots \end{cases}$$

Table 1: **Properties of the Continuous-Time Fourier Series**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)dt$ (finite-valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$

Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$