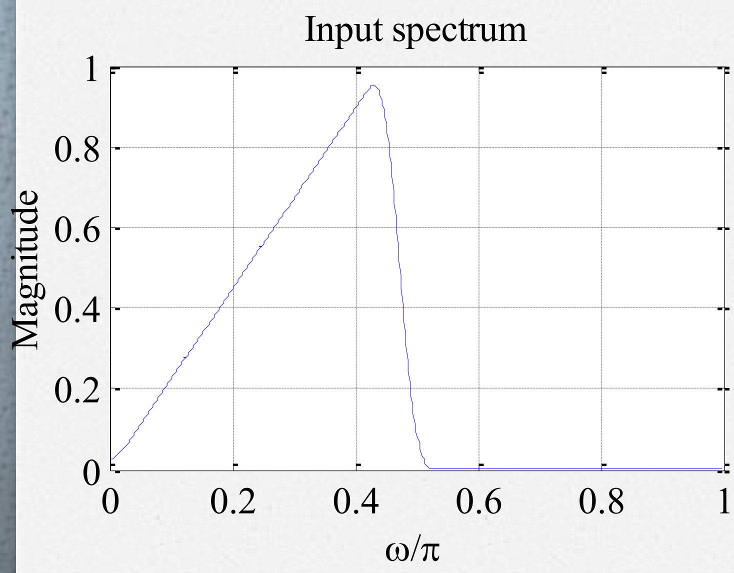
Digital Signal Processing- Lecture 21

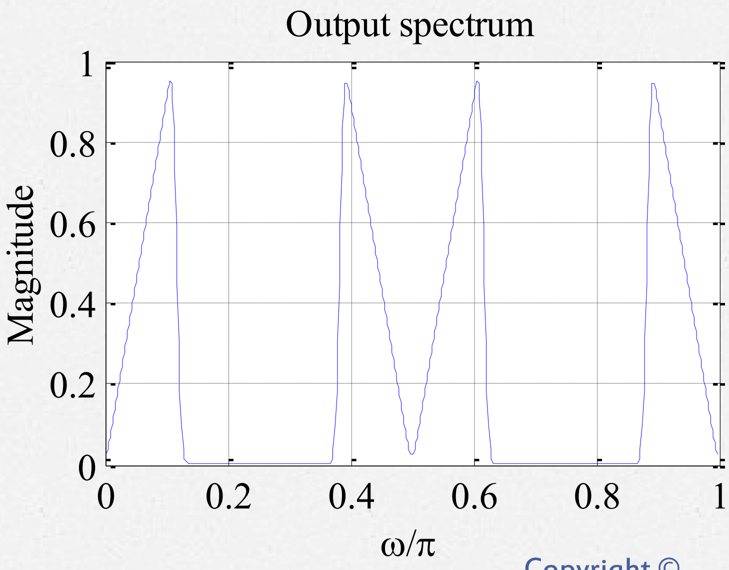
Topics to be covered:

- Up sampling

Up-Sampler

- Program 10_3 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $L = 4$





Down-Sampler

Frequency-Domain Characterization

- Applying the *z*-transform to the input-output relation of a factor-of-*M* down-sampler

$$y[n] = x[Mn]$$

we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n}$$

- The expression on the right-hand side cannot be directly expressed in terms of $X(z)$

Down-Sampler

- To get around this problem, define a new sequence :

$$x_{\text{int}}[n]$$

- Then

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_{\text{int}}[k] z^{-k/M} = X_{\text{int}}(z^{1/M})$$

Down-Sampler

- Taking the *z*-transform of $x_{int}[n] = c[n] \cdot x[n]$ and making use of

$$c[n] = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{kn}$$

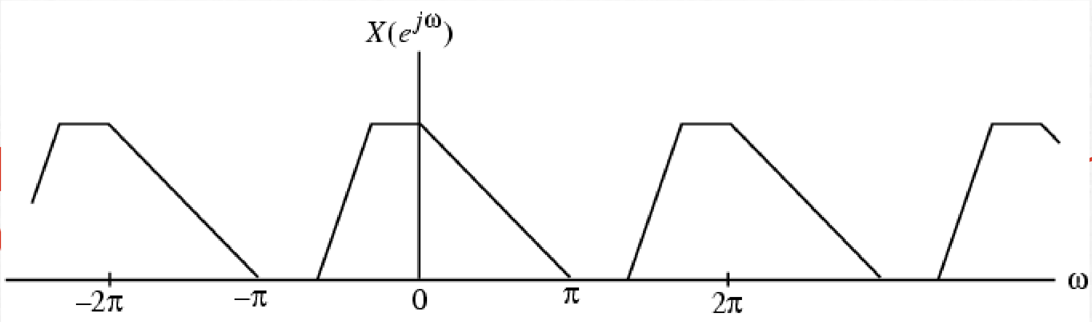
we arrive at

$$X_{int}(z) = \sum_{n=-\infty}^{\infty} c[n]x[n]z^{-n} = \frac{1}{M}\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{M-1} W_M^{kn}\right)x[n]z^{-n}$$

$$= \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{n=-\infty}^{\infty} x[n]W_M^{kn}z^{-n}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW_M^{-k}\right)$$

Down-Sampler

- Consider a factor-of-2 down-sampler with an input $x[n]$ whose spectrum is as shown below

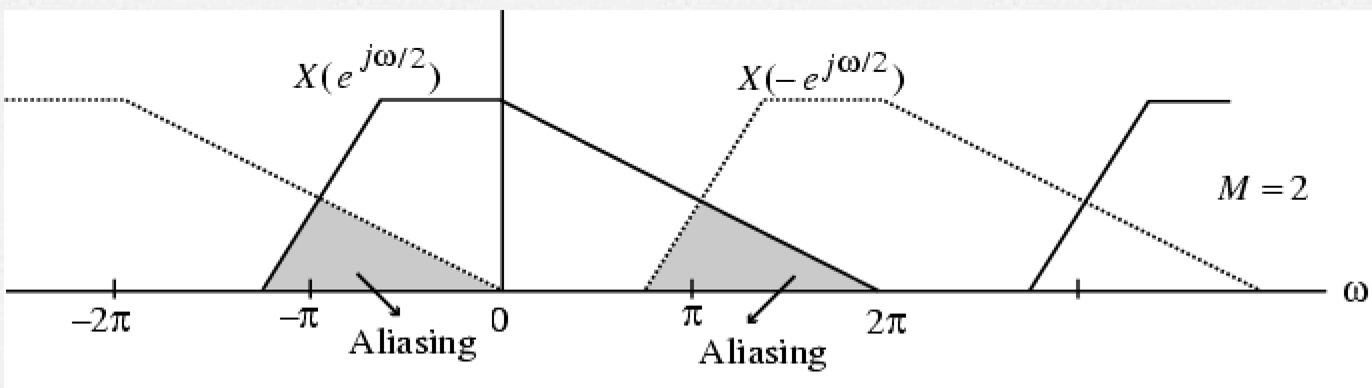


- The DT nces of this do

$$Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\}$$

Down-Sampler

- Now $X(-e^{j\omega/2}) = X(e^{j(\omega-2\pi)/2})$ implying that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below



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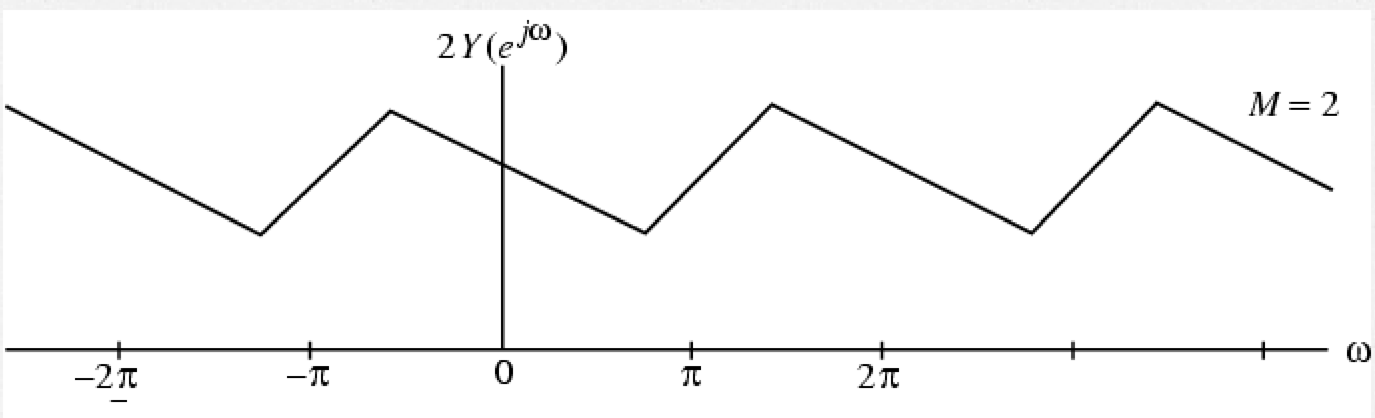
Down-Sampler

- The plots of the two terms have an overlap, and hence, in general, the original "*shape*" of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below



Down-Sampler

- This overlap causes the *aliasing* that takes place due to under-sampling
- There is no overlap, i.e., no aliasing, only if

$$X(e^{j\omega}) = 0 \qquad \text{for } |\omega| \geq \pi/2$$

- Note: $Y(e^{j\omega})$ is indeed periodic with a period 2π, even though the stretched version of $X(e^{j\omega})$ is periodic with a period 4π

Down-Sampler

- For the general case, the relation between the DTFTs of the output and the input of a factor-of-*M* down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

- $Y(e^{j\omega})$ is a sum of *M* uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of 1/*M*

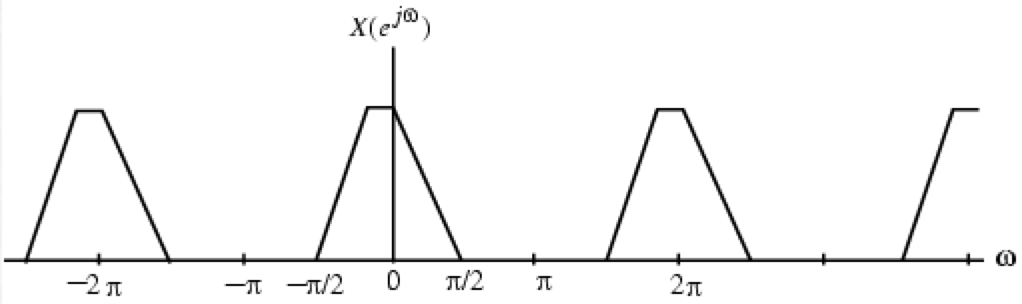
Down-Sampler

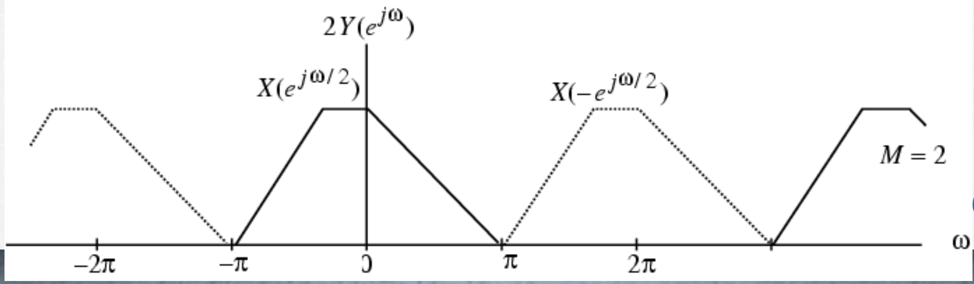
- Aliasing is absent if and only if

$$X(e^{j\omega}) = 0 \;\; for \;\; |\omega| \geq \pi / M$$

as shown below for $M = 2$

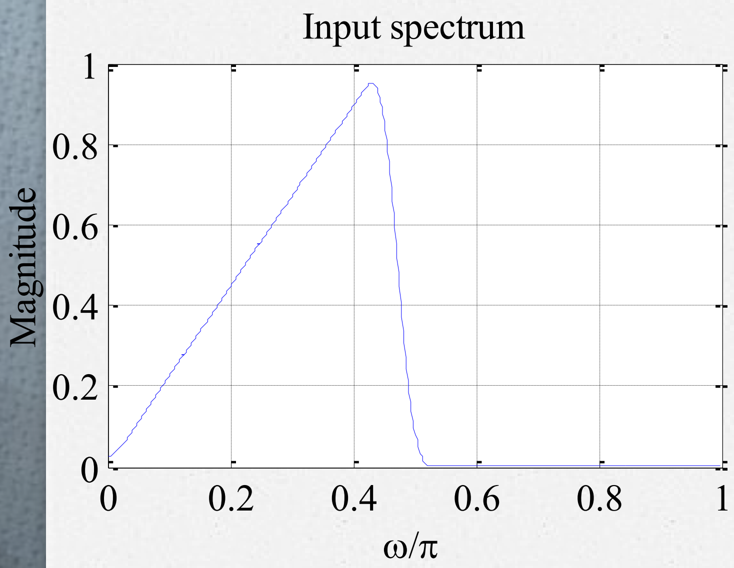
$$X(e^{j\omega}) = 0 \;\; \text{for} \;\; |\omega| \geq \pi / 2$$

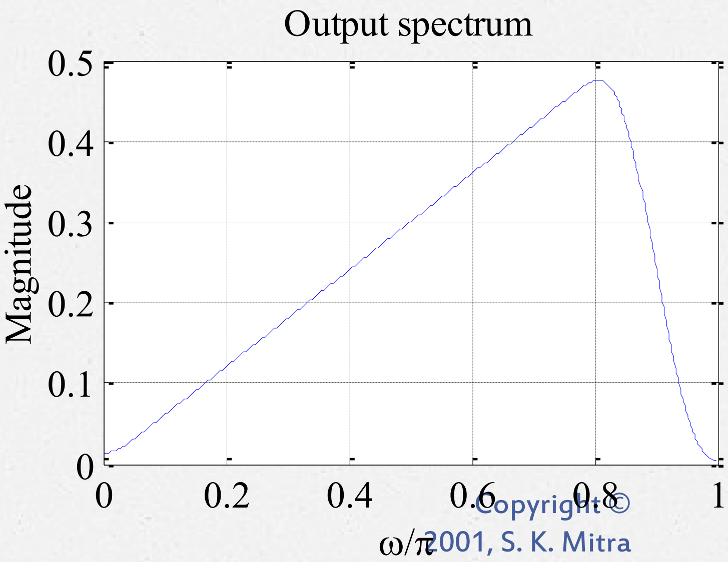




Down-Sampler

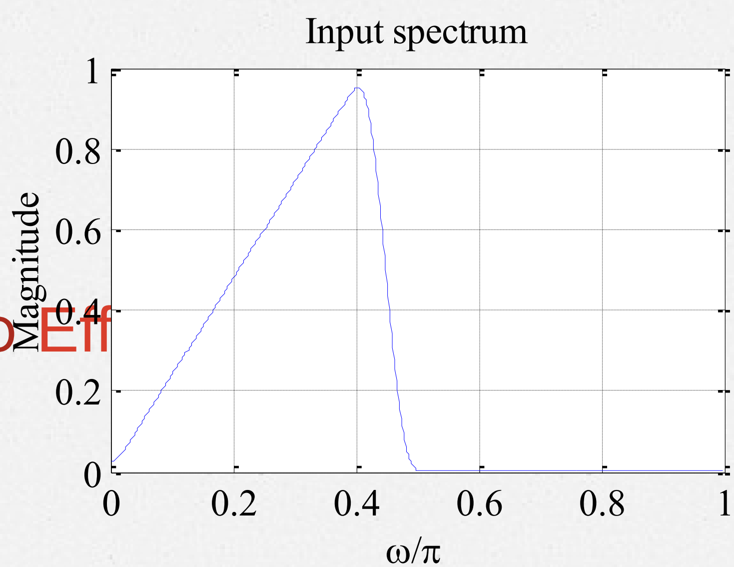
- Program 10_4 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $M = 2$

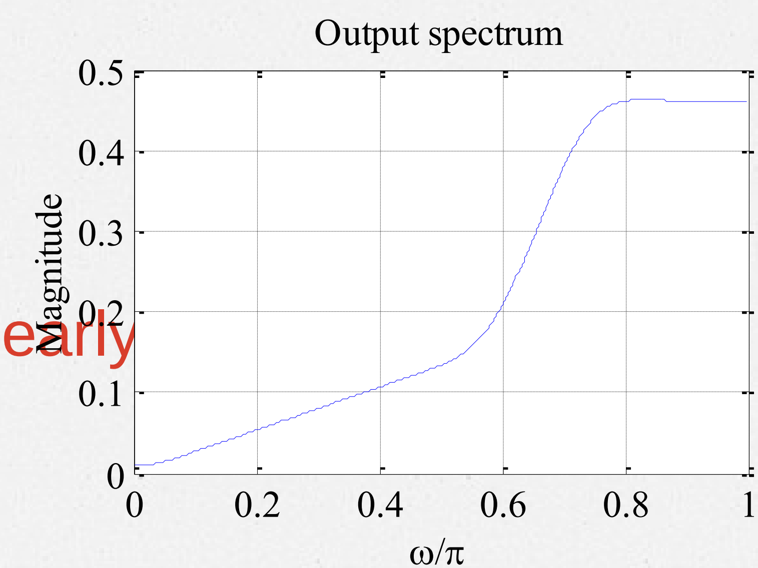




Down-Sampler

- The input and output spectra of a down-sampler with $M = 3$ obtained using Program 10-4 are shown below





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