Digital Signal Processing- Lecture 9

Topics to be covered:

- Z-transform

Ch. 11 Z-Transform & D-T Systems

Z-Transform does for DT systems what the Laplace Transform does for CT systems

Z-T is used to

Solve difference equations with…

Solve zero-state systems using…

We will:

- Define the ZT
- See its properties
- Use the ZT and its properties to analyze D-T systems

So for the Laplace transform we looked at: $s = \sigma + j\omega$ which is in rect. form

But, for Z-transform we use:

Q: Why the change?

A: Suffice to say…it has to do with the periodic nature of the DTFT.

Remember that the DTFT is a periodic function of Ω… and by using $z = \alpha e^{j\Omega}$ we stick Ω in as an angle which forces the periodic dependence on Ω.

Just like for Laplace… there are two forms of the Z-Transform:

Two sided Z-transform

$$X_2(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z \text{ is complex - valued}$$

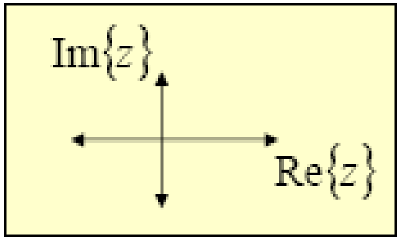
One sided Z-transform

$$X_1(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \qquad z \text{ is complex - valued}$$

If $x[n]$ is a causal signal: $X_1(z) = X_2(z)$

Our Focus is Here

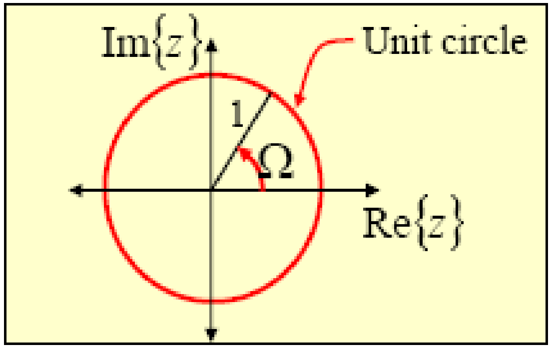
So… the Z-Transform gives a complex-valued function on the "z-plane"



Recall: for Laplace we had the *s*-plane… and we divided it into two parts:

- those values of *s* to the left of the *jω*-axis (left-half plane)
- those values of *s* to the right of the *jω*-axis (right-half plane)

"**<u>Unit Circle</u>**" = all *z* such that $|z| = 1$, i.e. all $z = e^{j\Omega}$



For the Z-Transform we'll need to divide the plane into parts:

- those values of *z* inside the unit circle
- those values of *z* outside the unit circle

Region of Convergence (ROC)

Set of all z values for which the sum in the ZT definition converges

Each signal has its own region of convergence.

(Same idea as for Laplace Transform)

Example of Finding the ZT: Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$Z\{\delta[n]\} = \sum_{n=0}^{\infty} \delta[n] z^{-n}$$

$$= \cdots + 0 \times z^{-1} + 1 \times z^{0} + 0 \times z^{1} + \cdots$$

$$= 1$$

$$\delta[n] \quad \leftrightarrow \quad 1$$

ROC = all complex #'s

This result and many others are on Table of Z Transforms available on my website… please it rather than the one in your book, which has some errors

Example of Finding the ZT: Unit Step $u[n]$

$$U(z)=\sum_{n=0}^{\infty}u[n]z^{-n}=\sum_{n=0}^{\infty}z^{-n}=\frac{z}{z-1}=\frac{1}{1-z^{-1}}$$

ROC =

Using standard result for "geometric sum"

$$u[n] \quad \leftrightarrow \quad \frac{z}{z-1}=\frac{1}{1-z^{-1}}$$

Example of Finding the ZT: Causal Exponential

$$x[n]=a^n u[n]$$

Again using geometric sum: $X(z)=\sum_{n=0}^{\infty}a^n z^{-n}=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n=\frac{z}{z-a}=\frac{1}{1-az^{-1}}$

ROC = all z such that |z| > |a|

$$a^n u[n] \quad \leftrightarrow \quad \frac{z}{z-a}=\frac{1}{1-az^{-1}}$$

Relationship between ZT & DTFT

Recall: for some signals the CTFT was embedded in the LT
(If the ROC includes the $j\omega$-axis)

We have a similar condition for the DTFT and the ZT…

If ROC includes the unit circle, then we can say that: $X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$

$X(\Omega)$ = "walk around the unit circle" and get $X(z)$ values

Explains why $X(\Omega)$ is periodic… Ω is an "angle around the unit circle"

⇒ Once we've walked around the unit circle… going farther just repeats the values $X(z)$ that we are grabbing

⇒ We only need to worry about $\Omega \in [-\pi$ to $\pi)$

7.3 Inverse Z-T

Same story as for LT: using the integral inversion formula is hard!

The use of partial fractions here is ***almost*** exactly the same as for Laplace transforms…

… the only difference is that you first divide by z ***before*** performing the partial fraction expansion… then after expanding you multiply by z to get the final expansion.

Example of Partial Fraction for Inverse ZT:

Suppose you want to find the inverse ZT of

$$Y(z) = \frac{z+1}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

First divide $Y(z)$ by z to get:

$$\frac{Y(z)}{z} = \frac{z+1}{z^3 + \frac{3}{4}z^2 + \frac{1}{8}z}$$

Then use matlab's residue to do a partial fraction expansion on $Y(z)/z$

```
[r,p,k]=residue([1 1],[1 0.75 0.125 0])

r =                 p =                 k = []
     4                 -0.5000
   -12                 -0.2500
     8                  0
```

Then we have:

$$\frac{Y(z)}{z} = \frac{4}{z+\frac{1}{2}} - \frac{12}{z+\frac{1}{4}} + \frac{8}{z} \quad \Rightarrow \quad Y(z) = \frac{4z}{z+\frac{1}{2}} - \frac{12z}{z+\frac{1}{4}} + 8$$

Now… the point of dividing by z becomes clear… you get terms like this (with z's in the numerator)… and they are on the ZT table!!!

$$\Rightarrow \quad y[n] = 4(-\tfrac{1}{2})^n u[n] - 12(-\tfrac{1}{8})^n u[n] + 8\delta[n]$$

11.2 Properties of ZT

Linearity: Same ideas as for CTFT, DTFT, and LT

Right Shift for Causal Signal

Let $x[n] = 0,\ n < 0$

If $x[n] \leftrightarrow X(z)$, then

"Proof": $X(z) = x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots$

$$Z\{x[n-q]\} = \underbrace{0z^{0} + 0z^{-1} + \ldots + 0z^{-q+1}}_{=0} + x[0]z^{-q} + x[1]z^{-q-1} + \ldots$$

$$= x[0]z^{0}z^{-q} + x[1]z^{-1}z^{-q} + x[2]z^{-2}z^{-q} + \ldots$$

Pull out the z^{-q}

$$= z^{-q}\underbrace{\left[x[0]z^{0} + x[1]z^{-1} + \ldots\right]}_{= X(z)}$$

Example of Applying the Right-Shift Property for Causal Signals

Suppose we want to find the Z-T of the pulse signal:

$$p[n] = \begin{cases} 1, \; n = 0, 1, 2, \ldots, q-1 \\ 0, \; else \end{cases}$$

Well.. We can write this pulse in terms of the unit step:

$$p[n] = u[n] - u[n-q]$$

Now, by linearity of the ZT we have: $P(z) = Z\{u[n]\} - Z\{u[n-q]\}$

But we already know that $Z\{u[n]\} = \dfrac{z}{z-1}$

Using the Right-Shift Property gives $Z\{u[n-q]\} = z^{-q}\dfrac{z}{z-1}$

So…

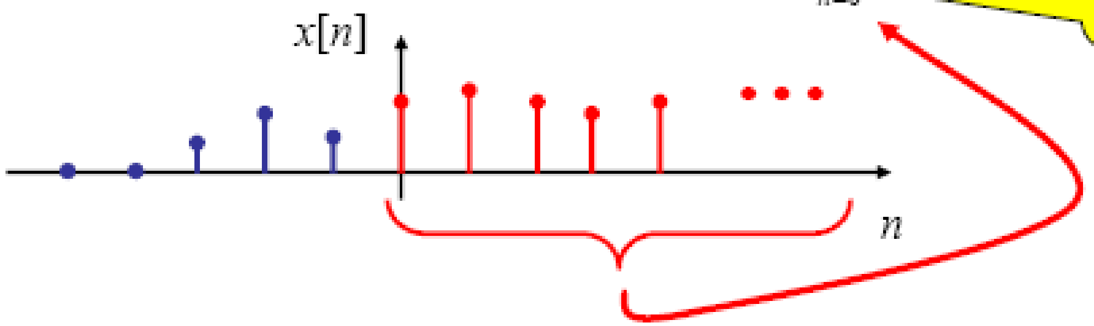
$$P(z) = \left[\frac{z}{z-1}\right] - z^{-q}\left[\frac{z}{z-1}\right] = \frac{z(1-z^{-q})}{z-1}$$

One-Sided ZT of the Right shift of *Non*-causal $x[n]$

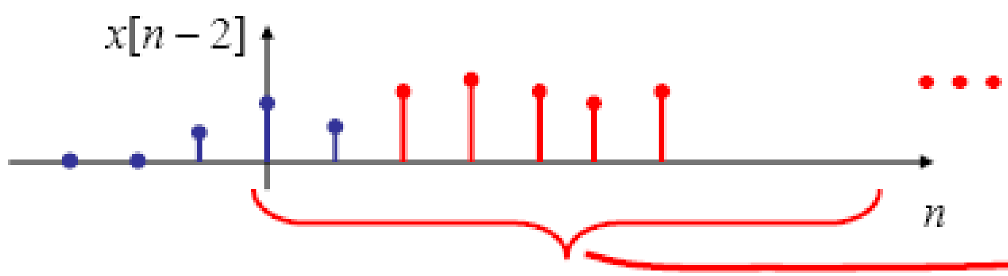
Let $x[n]$ be a non-causal signal… $x[n] \neq 0$ for some $n < 0$

Then the One-Side ZT is: $x[n] \quad \leftrightarrow \quad X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

Because this is the <u>One</u>-Sided ZT… not all non-zero values of $x[n]$ are used here!!!



Note that right-shifting a <u>non</u>-causal signal brings new values into the one-sided ZT summation!!!



$$Z\{x[n-2]\} = \sum_{n=0}^{\infty} x[n-2] z^{-n}$$

What is $Z\{x[n-q]\}$ in terms of $X(z)$??

We'll write this property for the first 2 values of q…

$$x[n-1] \quad \leftrightarrow \quad z^{-1}X(z)+x[-1]$$

$$x[n-2] \quad \leftrightarrow \quad z^{-2}X(z)+x[-1]z^{-1}+z[-2]$$

$$\vdots \qquad \qquad \vdots$$

… and then write the general result:

$$x[n-q] \quad \leftrightarrow \quad z^{-q}X(z)+x[-1]z^{-q+1}+x[-2]z^{-q+2}+\ldots+z^{-1}x[-q+1]+x[-q]$$

"Proof" for $q = 2$

$$Z\{x[n-q]\}=x[-2]z^{0}+x[-1]z^{-1} \quad + \quad x[0]z^{-2}+x[1]z^{-3}+\ldots$$

$$=x[-2]z^{0}+x[-1]z^{-1} \quad + \quad z^{-2}\left(x[0]z^{0}+x[1]z^{-1}+\ldots\right)$$

Parts that get "shifted into" the one-sided ZT's "machinery"

$X(z)$

Solving a First-order Difference Equation using the ZT

Given : $y[n] + ay[n-1] = x[n]$

$\text{IC} = y[-1]$

$x[n]$ for $n = 0, 1, 2, \ldots$

Solve for: $y[n]$ for $n = 0, 1, 2, \ldots$

Take ZT of differential equation: $Z\{y[n] + ay[n-1]\} = Z\{x[n]\}$

$$Z\{y[n]\} + aZ\{y[n-1]\} = Z\{x[n]\}$$

$Y(z)$

$X(z)$

Need Right-Shift Property…
but which one???

Because of the non-zero IC we need to use the *non*-causal form:

$$Z\{y[n-1]\} = z^{-1}Y(z) + y[-1]$$

Using these results gives: $$Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)$$

Which is an algebraic equation that can be solved for $Y(z)$:

$$Y(z) = \frac{-ay[-1]}{1+az^{-1}} + \frac{b}{1+az^{-1}}X(z)$$

Not the best form for doing Inverse ZT… we want things in terms of z not z^{-1}

$$Y(z) = -ay[-1]\frac{z}{z+a} + \frac{bz}{z+a}X(z)$$

On ZT Table

Part due to input signal modified by **Transfer Function**

$$H(z) = \frac{bz}{z+a}$$

$$y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\}$$

If $|a| < 1$ this dies out as $n \uparrow$, its an IC-driven transient

If the ICs are zero, this is all we have!!!