Digital Signal Processing- Lecture 11

Topics to be covered:

- Inverse Z transform

first order discrete system. Let $y(n)$ = money in an account at the start of the n-th compounding period , and let p be the interest rate (per compounding period) . If $x(n)$ is the deposit made at the start of the n-th period then the amount in the account at the start of the next period is given by

$$y(n) = (1+p)y(n-1) + x(n)$$

Assume that $y(n) = 0$ for $n < 0$ and $x(n) = 0$ for $n < 0$, then take the z-transform of this equation to get

$$Y(z) = (1+p)z^{-1}Y(z) + X(z) \Rightarrow Y(z) = \frac{X(z)}{1-az^{-1}} \quad \text{where } a = 1+p \text{ and } |z| > 1+p$$

Obviously an investor would rather know $y(n)$ than $Y(z)$.

There are three ways in which the z-transform is usually inverted, in order of increasing generality

(i). Long division
(ii). Partial fractions
(iii). Residues

Long Division Method

This relies directly on the definition of the z-transform and is useful if the first few terms in the sequence are required. The idea is to expand the z transform as power series in z and then use the definition to read off the successive values of the signal.

For example if , in the example above, only one deposit was made, at say $n = 0$, then

$$X(z) = x(0)$$

and

$$Y(z) = x(0)\frac{z}{z-a}$$

Then dividing the numerator into the denominator repeatedly yields

$$
\begin{array}{l}
\qquad 1 + az^{-1} + a^2 z^{-2} + \cdots \\
z - a \overline{)\, z \qquad\qquad\qquad\qquad} \\
\qquad \underline{z - a} \\
\qquad 0 + a \\
\qquad\quad \underline{a - a^2 z^{-1}} \\
\qquad\quad 0 + a^2 z^{-1} \\
\qquad\qquad \underline{a^2 z^{-1} - a^3 z^{-2}} \\
\qquad\qquad 0 \quad + \quad a^3 z^{-2} \\
\qquad\qquad\qquad \cdots
\end{array}
$$

so that $Y(z) = x(0)(z^{-0} + az^{-1} + a^2 z^{-2} + \cdots)$. Comparing this with the definition

$$Y(z) = y(0)z^{-0} + y(1)z^{-1} + y(2)z^{-2} + \cdots$$

leads to $y(0) = x(0)$, $y(1) = ax(0)$, $y(2) = a^2 x(0)$, $\cdots$ which is the well known compound interest result spelt out in laborious detail.

Inversion by Partial Fraction Expansion

Sequences with rational z-transforms are rather stereotyped, mostly involving combinations of a limited number of standard forms. This makes it feasible to invert the z-transform by expanding it as a sum of simple terms, each of which has a known inverse. Thus the inversion process is one of reduction to a standard form followed by table look-up and often copious algebra.

For example, consider a causal second order system

$$y(n) - 2a\cos\theta y(n-1) + a^2 y(n-2) = x(n)$$

has transfer function

$$H(z) = \frac{1}{1 - 2z^{-1}a\cos\theta + a^2 z^{-2}} = \frac{z^2}{z^2 - 2za\cos\theta + a^2} \quad : \ |z| > a$$

The first step is to reduce the ratio to a proper fraction, which is most conveniently done here by dividing both sides by z. Then the denominator is factored.

$$\frac{1}{z}H(z) = \frac{z}{(z - ae^{i\theta})(z - ae^{-i\theta})} \quad : \ |z| > a$$

Then we attempt to write the fraction in the form

$$\frac{z}{(z - ae^{i\theta})(z - ae^{-i\theta})} = \frac{C_1}{(z - ae^{i\theta})} + \frac{C_2}{(z - ae^{-i\theta})}$$

where C_1 and C_2 are constants. Note that this expression must hold for all z so we can choose any particular value of z that makes it easy to calculate the constants. Suppose we multiply both sides by $(z - ae^{i\theta})$

$$\frac{(z-ae^{i\theta})z}{(z-ae^{i\theta})(z-ae^{-i\theta})} = C_1 + \frac{(z-ae^{i\theta})C_2}{(z-ae^{-i\theta})}$$

The LHS of this expression simplifies, so

$$\frac{z}{(z-ae^{-i\theta})} = C_1 + \frac{(z-ae^{i\theta})C_2}{(z-ae^{-i\theta})}$$

and if we choose $z = ae^{i\theta}$ the second term disappears, provided ($e^{i\theta} \neq e^{-i\theta}$ ie. $e^{2i\theta} \neq 1$ or $\theta \neq k\pi$) and we get

$$C_1 = \frac{ae^{i\theta}}{a(e^{i\theta} - e^{-i\theta})} = \frac{e^{i\theta}}{2i\sin\theta}$$

Similarly multiplying both sides by $(z-ae^{-i\theta})$ and choosing $z = ae^{-i\theta}$ leads to

$$C_2 = \frac{ae^{-i\theta}}{a(e^{-i\theta} - e^{i\theta})} = \frac{-e^{-i\theta}}{2i\sin\theta}$$

Then $H(z) = \frac{1}{2i\sin\theta}\left(\frac{ze^{i\theta}}{(z-ae^{i\theta})} - \frac{ze^{-i\theta}}{(z-ae^{-i\theta})}\right) \ : \ |z| > a$

But, from the examples of the z-transform given earlier

$$u(n)\alpha^n \leftrightarrow \frac{z}{z-\alpha} \quad : \quad |z| > |\alpha|$$

so that

$$h(n) = \frac{1}{2i\sin\theta}\left(u(n)a^n e^{i(n+1)\theta} - u(n)a^n e^{-i(n+1)\theta}\right) = u(n)a^n \frac{\sin((n+1)\theta)}{\sin\theta} \quad : \quad \theta \neq k\pi$$

General method

The example above can be generalised to the case of proper rational fractions whose denominators do not have repeated roots, ie to cases where $H(z) = \frac{Q(z)}{P(z)}$ where Q and P are polynomials and the order of Q is less than that of P and the solutions of $P(z) = 0$ are all distinct The trick is to write $P(z) = \prod_{k=1}^{N} (z - p_k)$ (assume the coefficient of z^n in P is one).

Then try writing $H(z) = \frac{Q(z)}{\prod_{k=1}^{N} (z - p_k)} = \sum_{k=1}^{N} \frac{C_k}{(z - p_k)}$

Now $(z - p_n)H(z) = \frac{Q(z)}{\prod_{k \neq n}^{N} (z - p_k)} = C_n + (z - p_n) \sum_{k \neq n}^{N} \frac{C_k}{(z - p_k)}$

and if none of the denominators in the sum is equal to $(z - p_n)$ then if we put $(z = p_n)$ the last term is zero and then

$$C_n = \frac{Q(p_n)}{\prod_{k \neq n}^{N} (p_n - p_k)} \quad : \quad n = 1,2,3 \cdots, N.$$

Non repeated root examples

The third order causal system

$$y(n) - 0.25y(n-1) + 0.25y(n-2) - 0.0625y(n-3) = x(n)$$

has the transfer function

$$H(z) = \frac{1}{1 - 0.25z^{-1} + 0.25z^{-2} - 0.0625z^{-3}} \quad : \quad |z| > 0.5$$

which is equivalent to

$$H(z) = \frac{z^3}{z^3 - 0.25z^2 + 0.25z - 0.0625}$$

Reduce this to a proper fraction by dividing both sides by z and factor the denominator (this usually needs to be done numerically),

$$\frac{1}{z}H(z) = \frac{z^2}{(z - i0.5)(z + i0.5)(z - 0.25)}$$

Since the roots are all distinct the PFE of this is

The Impuse response is:

$$h(n) = 0.447e^{-i0.4636}(0.5i)^n + 0.447e^{i0.4636}(-0.5i)^n + 0.8(0.25)^n \quad : \quad n \geq 0$$

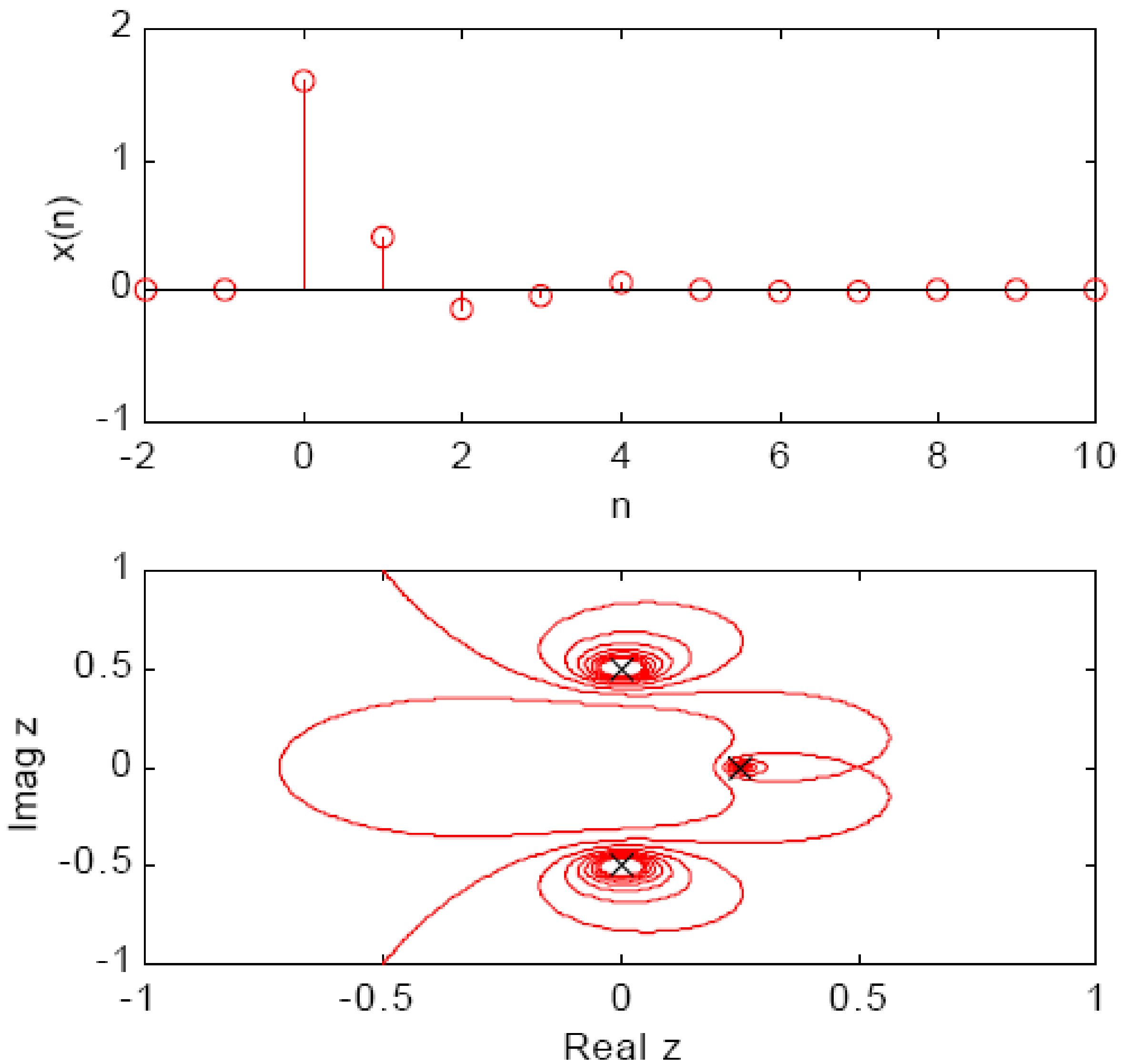
This expression is real since the second term is the complex conjugate of the first. It can be simplified a little by writing

$$i^n = e^{in\pi/2}$$

whence

$$h(n) = 2\,\mathrm{Re}\left[e^{i(n\pi/2-0.4636)}\right]0.447(0.5)^n + 0.8(0.25)^n$$

$$= \frac{0.894\cos(n\pi/2 - 0.4636)}{2^n} + \frac{0.8}{2^{2n}} \quad : \quad n \geq 0$$



2
1
0
-1
x(n)
-2
0
2
4
6
8
10
n
1
0.5
0
-0.5
-1
Imag z
-1
-0.5
0
0.5
1
Real z

Rational systems with repeated

$$X(z) = \frac{1}{(z-1)^2(z-2)}$$

the PFE is taken to be of the form

$$X(z) = \frac{C_1}{(z-2)} + \frac{C_{21}}{(z-1)} + \frac{C_{22}}{(z-1)^2}$$

The coefficients C_1 and C_{22} can be calculated using the usual trick: Multiply both sides by $(z-2)$ to get

$$(z-2)X(z) = \frac{1}{(z-1)^2} = C_1 + \frac{(z-2)C_{21}}{(z-1)} + \frac{(z-2)C_{22}}{(z-1)^2}$$

and then set $z = 2$ to get

$$C_1 = \frac{1}{(2-1)^2} = 1$$

$$\frac{d\left[(z-1)^2 X(z)\right]}{dz} = \frac{-1}{(z-2)^2} = \frac{d}{dz}\left[\frac{(z-1)^2 C_1}{(z-2)} + (z-1)C_{21} + C_{22}\right]$$

$$= \frac{(z-1)\left[2(z-2)-(z-1)\right]C_1}{(z-2)^2} + C_{21}$$

and then take $z=1$

$$C_{21} = \left.\frac{d\left[(z-1)^2 X(z)\right]}{dz}\right|_{z=1} = \frac{-1}{(1-2)^2} = -1$$

Thus the final PFE is

$$X(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)} - \frac{1}{(z-1)^2}$$

which may be checked by reducing the expression to a common denominator.

The general case of a k-th order pole is easily, if tediously, handled by similar methods. If one of the poles (say p_m) in the proper fraction $X(z)$ is of k-th order then

$$X(z) = \frac{Q(z)}{(z-p_m)^k \widetilde{P}(z)}$$

where $\widetilde{P}(z)$ is a polynomial whose roots do not include p_m.

The PFE is then written

$$X(z) = \frac{C_{m1}}{(z-p_m)} + \frac{C_{m1}}{(z-p_m)^2} + \cdots + \frac{C_{mk}}{(z-p_m)^k} + A(z)$$

where A(z) contains terms which do not involve p_m. Multiply by $(z-p_m)^k$

$$(z-p_m)^k X(z) = \frac{Q(z)}{\widetilde{P}(z)} = C_{m1}(z-p_m)^{k-1} + C_{m2}(z-p_m)^{k-2} + \cdots + C_{mk} + (z-p_m)^k A(z)$$

differentiate n times $n < k$ then set $z = p_m$ to see that

$$\frac{d^n}{dz^n}\left[(z - p_m)^k X(z)\right]\Bigg|_{z=p_m} = \frac{d^n}{dz^n}\left[\frac{Q(z)}{\tilde{P}(z)}\right]\Bigg|_{z=p_m} = n!C_{m(k-n)}$$

The expression on the right will be finite because all the poles at $z = p_m$ have been removed. (The term in A(z) can be also shown not to contribute to the final expression since it contains no poles at $z = p_m$.)

Thus the set of PFE coefficients for a pole of order k are given by

$$C_{m(k-n)} = \frac{1}{n!}\frac{d^n}{dz^n}\left[(z-p_m)^k X(z)\right]\Big|_{z=p_m} \quad : \; n = 0,1,2,\cdots,k-1.$$

(Note : the zeroth derivative of a function is just the function itself.)

Repeated roots PFE Example

Consider the causal system described by the difference equation

$$y(n) - 5y(n-1) + 9y(n-2) - 7y(n-3) + 2y(n-4) = x(n)$$

The transfer function is

$$H(z) = \frac{1}{1 - 5z^{-1} + 9z^{-2} - 7z^{-3} + 2z^{-4}} = \frac{z^4}{z^4 - 5z^3 + 9z^2 - 7z + 2}$$

Fortunately the denominator factors, thus :

$$H(z) = \frac{z^4}{(z-1)^3(z-2)}$$

and it is clear that for a causal system the ROC is $|z| > 2$ and the system is unstable.
We reduce the RHS to a proper fraction by dividing by z. Then the PFE is of the form

$$\frac{1}{z}H(z) = \frac{C_{11}}{(z-1)} + \frac{C_{12}}{(z-1)^2} + \frac{C_{13}}{(z-1)^3} + \frac{C_2}{(z-2)}$$

Using the non-repeated roots expression for the coefficient C_2 gives

$$C_2 = \left[(z-2)\frac{1}{z}H(z)\right]_{z=2} = \left[\frac{z^3}{(z-1)^3}\right]_{z=2} = 8$$

The repeated roots expression is then used for the other three coefficients

$$C_{13} = (z-1)^3 \frac{1}{z}H(z)\bigg|_{z=1} = \frac{z^3}{(z-2)}\bigg|_{z=1} = -1$$