Digital Signal Processing- Lecture 10

Topics to be covered:

- Inverse Z transform

first order discrete system. Let $y(n)$ = money in an account at the start of the n-th compounding period , and let p be the interest rate (per compounding period) . If $x(n)$ is the deposit made at the start of the n-th period then the amount in the account at the start of the next period is given by

$$y(n) = (1+p)y(n-1) + x(n)$$

Assume that $y(n) = 0$ for $n < 0$ and $x(n) = 0$ for $n < 0$, then take the z-transform of this equation to get

$$Y(z) = (1+p)z^{-1}Y(z) + X(z) \Rightarrow Y(z) = \frac{X(z)}{1-az^{-1}} \text{ where } a = 1+p \text{ and } |z| > 1+p$$

Obviously an investor would rather know $y(n)$ than $Y(z)$.

There are three ways in which the z-transform is usually inverted, in order of increasing generality

(i). Long division
(ii). Partial fractions
(iii). Residues

Long Division Method

This relies directly on the definition of the z-transform and is useful if the first few terms in the sequence are required. The idea is to expand the z transform as power series in z and then use the definition to read off the successive values of the signal.

For example if , in the example above, only one deposit was made, at say $n = 0$, then

$$X(z) = x(0)$$

and

$$Y(z) = x(0)\frac{z}{z-a}$$

Then dividing the numerator into the denominator repeatedly yields

$$
\begin{array}{l}
\qquad\quad 1 + az^{-1} + a^2z^{-2} + \cdots \\
z - a)\,\overline{z \qquad\qquad\qquad\qquad\qquad} \\
\qquad\quad \underline{z - a} \\
\qquad\quad 0 + a \\
\qquad\qquad \underline{a - a^2z^{-1}} \\
\qquad\qquad 0 + a^2z^{-1} \\
\qquad\qquad\quad \underline{a^2z^{-1} - a^3z^{-2}} \\
\qquad\qquad\quad 0 \quad + \quad a^3z^{-2} \\
\qquad\qquad\qquad\quad \cdots
\end{array}
$$

so that $Y(z) = x(0)(z^{-0} + az^{-1} + a^2z^{-2} + \cdots)$. Comparing this with the definition

$$Y(z) = y(0)z^{-0} + y(1)z^{-1} + y(2)z^{-2} + \cdots$$

leads to $y(0) = x(0),\ \ y(1) = ax(0),\ \ y(2) = a^2x(0),\ \ \cdots$ which is the well known compound interest result spelt out in laborious detail.

Inversion by Partial Fraction Expansion

Sequences with rational z-transforms are rather stereotyped, mostly involving combinations of a limited number of standard forms. This makes it feasible to invert the z-transform by expanding it as a sum of simple terms, each of which has a known inverse. Thus the inversion process is one of reduction to a standard form followed by table look-up and often copious algebra.

For example, consider a causal second order system

$$y(n) - 2a\cos\theta y(n-1) + a^2 y(n-2) = x(n)$$

has transfer function

$$H(z) = \frac{1}{1 - 2z^{-1}a\cos\theta + a^2 z^{-2}} = \frac{z^2}{z^2 - 2za\cos\theta + a^2} \quad : \ |z| > a$$

The first step is to reduce the ratio to a proper fraction, which is most conveniently done here by dividing both sides by z. Then the denominator is factored.

$$\frac{1}{z}H(z) = \frac{z}{(z - ae^{i\theta})(z - ae^{-i\theta})} \quad : \ |z| > a$$

Then we attempt to write the fraction in the form

$$\frac{z}{(z - ae^{i\theta})(z - ae^{-i\theta})} = \frac{C_1}{(z - ae^{i\theta})} + \frac{C_2}{(z - ae^{-i\theta})}$$

where C_1 and C_2 are constants. Note that this expression must hold for all z so we can choose any particular value of z that makes it easy to calculate the constants. Suppose we multiply both sides by $(z - ae^{i\theta})$

$$\frac{(z-ae^{i\theta})z}{(z-ae^{i\theta})(z-ae^{-i\theta})} = C_1 + \frac{(z-ae^{i\theta})C_2}{(z-ae^{-i\theta})}$$

The LHS of this expression simplifies, so

$$\frac{z}{(z-ae^{-i\theta})} = C_1 + \frac{(z-ae^{i\theta})C_2}{(z-ae^{-i\theta})}$$

and if we choose $z = ae^{i\theta}$ the second term disappears, provided ($e^{i\theta} \neq e^{-i\theta}$ ie. $e^{2i\theta} \neq 1$ or $\theta \neq k\pi$) and we get

$$C_1 = \frac{ae^{i\theta}}{a(e^{i\theta} - e^{-i\theta})} = \frac{e^{i\theta}}{2i\sin\theta}$$

Similarly multiplying both sides by $(z-ae^{-i\theta})$ and choosing $z = ae^{-i\theta}$ leads to

$$C_2 = \frac{ae^{-i\theta}}{a(e^{-i\theta} - e^{i\theta})} = \frac{-e^{-i\theta}}{2i\sin\theta}$$

Then $H(z) = \frac{1}{2i\sin\theta}\left(\frac{ze^{i\theta}}{(z-ae^{i\theta})} - \frac{ze^{-i\theta}}{(z-ae^{-i\theta})}\right) \; : \; |z| > a$

But, from the examples of the z-transform given earlier

$$u(n)\alpha^n \leftrightarrow \frac{z}{z-\alpha} \quad : \quad |z| > |\alpha|$$

so that

$$h(n) = \frac{1}{2i\sin\theta}\left(u(n)a^n e^{i(n+1)\theta} - u(n)a^n e^{-i(n+1)\theta}\right) = u(n)a^n \frac{\sin((n+1)\theta)}{\sin\theta} \quad : \quad \theta \neq k\pi$$

General method

The example above can be generalised to the case of proper rational fractions whose denominators do not have repeated roots, ie to cases where $H(z) = \frac{Q(z)}{P(z)}$ where Q and P are polynomials and the order of Q is less than that of P and the solutions of $P(z) = 0$ are all distinct The trick is to write $P(z) = \prod_{k=1}^{N}(z - p_k)$ (assume the coefficient of z^n in P is one).

Then try writing $H(z) = \frac{Q(z)}{\prod_{k=1}^{N}(z - p_k)} = \sum_{k=1}^{N} \frac{C_k}{(z - p_k)}$

Now $(z - p_n)H(z) = \frac{Q(z)}{\prod_{k \neq n}^{N}(z - p_k)} = C_n + (z - p_n)\sum_{k \neq n}^{N} \frac{C_k}{(z - p_k)}$

and if none of the denominators in the sum is equal to $(z - p_n)$ then if we put $(z = p_n)$ the last term is zero and then

$$C_n = \frac{Q(p_n)}{\prod_{k \neq n}^{N}(p_n - p_k)} \quad : \quad n = 1,2,3 \cdots, N.$$

Non repeated root examples

The third order causal system

$$y(n) - 0.25y(n-1) + 0.25y(n-2) - 0.0625y(n-3) = x(n)$$

has the transfer function

$$H(z) = \frac{1}{1 - 0.25z^{-1} + 0.25z^{-2} - 0.0625z^{-3}} \quad : \quad |z| > 0.5$$

which is equivalent to

$$H(z) = \frac{z^3}{z^3 - 0.25z^2 + 0.25z - 0.0625}$$

Reduce this to a proper fraction by dividing both sides by z and factor the denominator (this usually needs to be done numerically),

$$\frac{1}{z}H(z) = \frac{z^2}{(z - i0.5)(z + i0.5)(z - 0.25)}$$

Since the roots are all distinct the PFE of this is