Digital Signal Processing- Lecture 6

Topics to be covered:

- IIR Filter Designing

Design of IIR Filters from Analog Filters

Objective: Design digital IIR filters using known analog filter results

- Analog filter design is mature and well understood

Recall: Analog systems are defined by

$$H_a(s) \;=\; \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt \qquad \text{[Laplace Trans.]}$$

Or, noting that s is the differentiation operator,

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \qquad \text{[Differential Equation]}$$

Approach: Map analog filters (s–plane) to digital filters (z–plane)

- Frequency Mapping: s–domain $j\Omega$ axis $\rightarrow$ z–domain unit circle
- Stability Mapping: s–domain LHP $\rightarrow$ inside z–domain unit circle

Aside: Recall linear phase in FIR filters requires

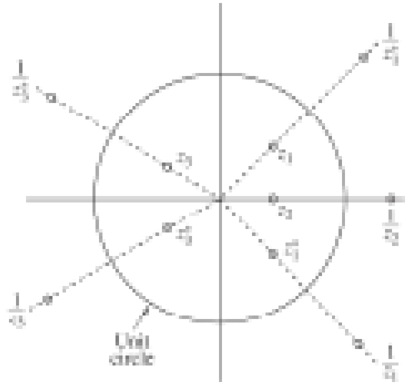
$$h(n) = \pm h(M-1-n) \quad \text{which implies} \quad z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

Thus roots of $H(z^{-1})$ and $H(z)$ are identical
$\Rightarrow$ roots are in reciprocal (complex–conj.) pairs

Note: The $z^{-(M-1)}H(z^{-1}) = \pm H(z)$ equality can be shown to be a necessary condition for linear phase

Observation: If $z^{-(M-1)}H(z^{-1}) = \pm H(z)$ in the IIR case $\Rightarrow$ poles and zeroes are in reciprocal (complex–conj.) pairs, i.e., there must be poles outside the unit circle

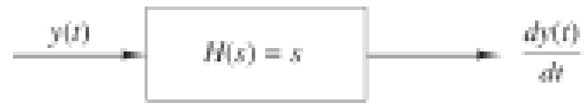
Result: Causal IIR filters can not have linear phase



Zero configuration for linear phase FIR filter

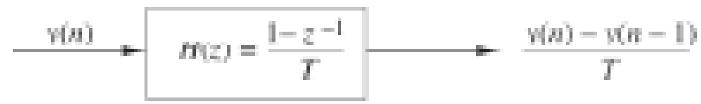
IIR Filter Design by Approximation of Derivatives

Approach: Since s it is the differentiation operator, define a discrete–time approximation



Continuous–time differentiator

Utilize the backward difference as a derivative approximation

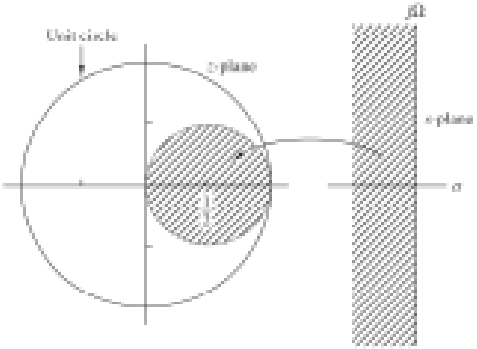


Discrete–time backward difference difference differentiator

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT - T)}{T} = \frac{y(n) - y(n-1)}{T}$$

Which has the system function $H(z) = \frac{1-z^{-1}}{T}$. Equating the operations

$$s = \frac{1 - z^{-1}}{T} \qquad \text{or} \qquad H(z) = H_a(s)\big|_{s=\frac{1-z^{-1}}{T}}$$



$z = \frac{1}{1 - j\Omega T}$ induced s–plane to z–plane mapping

Observations:

- Mapping yields stable filters – LHP of the s–plane → inside the z–plane unit circle
- Image inside the unit circle is in right half
  ⇒ *No high pass filters*

Example

Convert the analog bandpass filter

$$H_a(s) = \frac{1}{(s+0.1)^2+9}$$

into a digital filter using backward difference derivative approximation.

Utilizing $s = \frac{1-z^{-1}}{T}$ in the above

$$\begin{aligned} H(z) &= \frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9} \\ &= \frac{T^2/(1+0.2T+9.01T^2)}{1-\frac{2(1+0.1T)}{1+0.2T+9.01T^2}z^{-1}+\frac{1}{1+0.2T+9.01T^2}z^{-2}} \end{aligned}$$

Given T, this reduces a simple system

$$H(z) = \frac{1}{1+a_1z^{-1}+a_2z^{-2}}$$

IIR Filter Design by Impulse Invariance

Objective: Design a discrete–time IIR filter by sampling the impulse response of a continuous filter

Thus if $h_a(t)$ is a continuous–time impulse response, set

$$h(n) = h_a(nT), \quad n = 0, 1, \ldots$$

Consider the sampling induced frequency domain relation

$$\begin{aligned} H_a(s) &= \int_0^{\infty} h_a(t) e^{-st} dt \qquad \text{[Laplace Trans.]} \\ &= \sum_{n=0}^{\infty} h_a(nT) e^{-snT} \\ &= \sum_{n=0}^{\infty} h(n) e^{-snT} = H(z)|_{z=e^{sT}} \end{aligned}$$

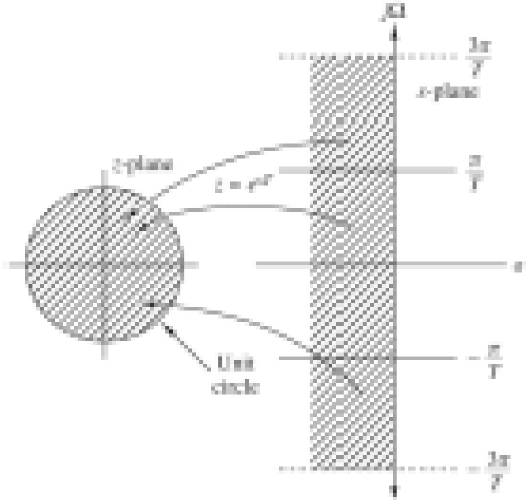
Result: This defines the the s–plane to z–plane mapping $z = e^{sT}$

Question: What s–plane $\leftrightarrow$ z–plane mapping does this perform?

Utilize $z = re^{j\omega}$ and $s = \sigma + j\Omega$ representations in the mapping

$$\begin{aligned} z &= e^{sT} \\ re^{j\omega} &= e^{\sigma T} e^{j\Omega T} \end{aligned}$$

$$\Rightarrow r = e^{\sigma T} \quad \text{and} \quad \omega = \Omega T$$



$z = e^{sT}$ induced s–plane to z–plane mapping

Observations:

- Mapping introduces aliasing (i.e., s and $(s + jk2\pi/T) \rightarrow$ same z)
- Zeros and poles don't follow the same mapping

IIR Filter Design by the Bilinear Transformation

Objective: Design a discrete–time IIR filter by approximating the integral (rather than the derivative, as before)

Suppose

$$H(s) = \frac{b}{s+a} \qquad \Rightarrow \dot{y}(t) + ay(t) = bx(t)$$

where $\dot{y} = \frac{dy(t)}{dt}$. Also note that

$$y(t) = \int_{-\infty}^{t} \dot{y}(\tau)d\tau = \int_{t_0}^{t} \dot{y}(\tau)d\tau + y(t_0)$$

Let $t = nT$ and $t_0 = (n-1)T$. Then (for T small)

$$\begin{aligned} \int_{t_0}^{t} \dot{y}(\tau)d\tau &\approx T\frac{(\dot{y}(t_0) + \dot{y}(t))}{2} \\ &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) \end{aligned}$$

$$\int_{t_0}^{t} \dot{y}(\tau)d\tau \quad \rightarrow \quad \text{area under } \dot{y}(t) \text{ between } t_0 \text{ and } t$$

$$\frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) \quad \rightarrow \quad \text{average of range start \& end points} \times \text{width}$$

Result: Approximately equal for small T

Thus

$$\begin{aligned} y(t) &= \int_{t_0}^{t} \dot{y}(\tau)d\tau + y(t_0) \\ \Rightarrow y(n) &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) + y(n-1) \qquad (*) \end{aligned}$$

We need an expression for $\dot{y}(n)$ in $(*)$. Recall $H(s) = \frac{b}{s+a}$ gives

$$\begin{aligned} \dot{y}(t) + ay(t) &= bx(t) \\ \Rightarrow \dot{y}(n) &= -ay(n) + bx(n) \qquad (**) \end{aligned}$$

Substitute $(**)$ into $(*)$

$$
\begin{aligned}
y(n) &= \frac{T}{2}(\dot{y}(n) + \dot{y}(n-1)) + y(n-1) \\
&= \frac{T}{2}([-ay(n) + bx(n)] + [-ay(n-1) + bx(n-1)]) + y(n-1)
\end{aligned}
$$

Rearranging, and then taking the z–transform, gives

$$
\begin{aligned}
\left(1 + \frac{Ta}{2}\right) y(n) - \left(1 - \frac{Ta}{2}\right) y(n-1) &= \frac{Tb}{2}(x(n) + x(n-1)) \\
Y(z)\left(1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}\right) &= X(z)\frac{bT}{2}(1 + z^{-1}) \\
\Rightarrow H(z) = \frac{Y(z)}{X(z)} &= \frac{\frac{bT}{2}(1 + z^{-1})}{1 + \frac{aT}{2} - (1 - \frac{aT}{2})z^{-1}}
\end{aligned}
$$

The denominator needs to be rearranged to lend insight

The denominator is

$$
\begin{aligned}
1 + \frac{aT}{2} - (1 - \frac{aT}{2})z^{-1} + \left[\frac{aT}{2}z^{-1} - \frac{aT}{2}z^{-1}\right] \\
&= 1 - \frac{aT}{2}z^{-1} - (1 - \frac{aT}{2})z^{-1} + \frac{aT}{2}(1 + z^{-1}) \\
&= 1 - z^{-1} + \frac{aT}{2}(1 + z^{-1})
\end{aligned}
$$

Thus $H(z)$ is expressed as

$$
\begin{aligned}
H(z) &= \frac{b\frac{T}{2}(1 + z^{-1})}{(1 - z^{-1}) + \frac{aT}{2}(1 + z^{-1})} \\
&= \frac{b}{\frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}}) + a}
\end{aligned}
$$

$$
\text{Recall } H(s) = \frac{b}{s + a} \Rightarrow \quad = H(s)|_{s=\frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})}
$$

Result: The mapping $s = \frac{2}{T}(\frac{1-z^{-1}}{1+z^{-1}})$ defines the bilinear transformation