Digital Signal Processing- Lecture 2

Topics to be covered:

- **Fourier Transform & its properties**
- **Discrete time random signal**

Fourier Transform Properties and Examples

- **Objectives**:

1. Properties of a Fourier transform
   - **Linearity** & **time shifts**
   - **Differentiation**
   - **Convolution** in the frequency domain

- **Background**
- While the **Fourier series/transform** is very important for representing a signal in the **frequency domain**, it is also important for **calculating a system's response (**convolution).
- A **system's transfer function** is the **Fourier transform** of its **impulse response**
- Fourier transform of a signal's **derivative** is **multiplication** in the **frequency domain**: $j\omega X(j\omega)$
- Convolution in the time domain is given by **multiplication** in the **frequency domain** (similar idea to log transformations)

- A CT signal $x(t)$ and its frequency domain, Fourier transform signal, $X(j\omega)$, are related by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ *analysis*

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$ *synthesis*

- This is denoted by:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

- For example:

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

- Often you have tables for common Fourier transforms
- The Fourier transform, $X(j\omega)$, represents the **frequency content** of $x(t)$.
- It exists either when $x(t)$->0 as $|t|$->∞ or when $x(t)$ is periodic (it generalizes the Fourier series)

Linearity of the Fourier Transform

- The Fourier transform is a **linear function** of *x*(*t*)

$$x_1(t) \overset{F}{\leftrightarrow} X_1(j\omega)$$

$$x_2(t) \overset{F}{\leftrightarrow} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \overset{F}{\leftrightarrow} aX_1(j\omega) + bX_2(j\omega)$$

- This follows directly from the definition of the Fourier transform (as the integral operator is linear) & it easily extends to an arbitrary number of signals
- Like impulses/convolution, if we know the Fourier transform of simple signals, we can calculate the Fourier transform of more complex signals which are a linear combination of the simple signals

Fourier Transform of a Time Shifted Signal

- We'll show that a Fourier transform of a signal which has a **simple time shift** is:

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

- i.e. the original Fourier transform but **shifted in phase** by $-\omega t_0$
- **Proof**
- Consider the Fourier transform synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right)e^{j\omega t}d\omega$$

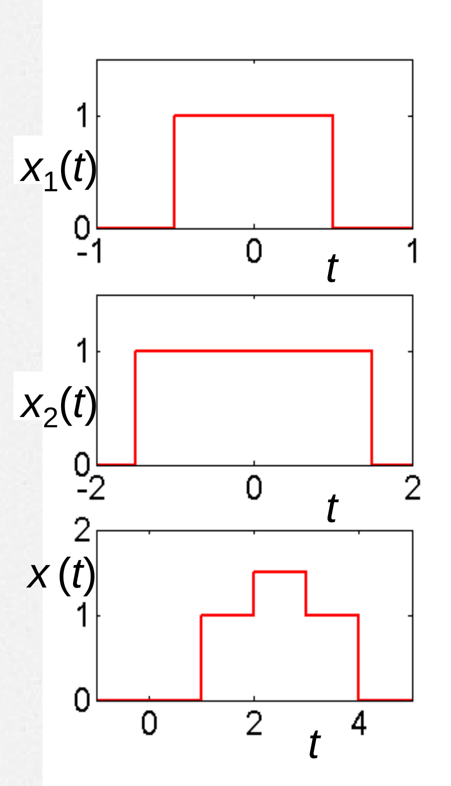
- but this is the synthesis equation for the Fourier transform
- $e^{-j\omega_0 t}X(j\omega)$

Example: Linearity & Time Shift

- Consider the signal (linear sum of two time shifted rectangular pulses)

$$x(t) = 0.5x_1(t-2.5) + x_2(t-2.5)$$

- where $x_1(t)$ is of width 1, $x_2(t)$ is of width 3, centred on zero (see figures)



- Using the FT of a rectangular pulse L10S7

$$X_1(j\omega) = 2\sin(\omega/2)/\omega$$

$$X_2(j\omega) = 2\sin(3\omega/2)/\omega$$

- Then using the **linearity** and **time shift** Fourier transform properties

$$X(j\omega) = e^{-j5\omega/2}\left(\left(\sin(\omega/2) + 2\sin(3\omega/2)\right)/\omega\right)$$

Fourier Transform of a Derivative

- By differentiating both sides of the Fourier transform synthesis equation with respect to *t*:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

- Therefore noting that this is the synthesis equation for the Fourier transform $j\omega X(j\omega)$

$$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

- This is very important, because it replaces **differentiation** in the **time domain** with **multiplication** (by $j\omega$) in the **frequency domain**.
- We can **solve ODEs** in the **frequency domain** using **algebraic** operations (see next slides)

Convolution in the Frequency Domain

- We can easily solve ODEs in the frequency domain:
  $$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$
- Therefore, to apply **convolution in the frequency domain**, we just have to **multiply** the **two Fourier Transforms**.
- To solve for the differential/convolution equation using Fourier transforms:

1. Calculate **Fourier transforms** of $x(t)$ and $h(t)$: $X(j\omega)$ by $H(j\omega)$
2. **Multiply** $H(j\omega)$ by $X(j\omega)$ to obtain $Y(j\omega)$
3. Calculate the **inverse Fourier transform** of $Y(j\omega)$

- $H(j\omega)$ is the LTI system's **transfer function** which is the **Fourier transform** of the **impulse response**, $h(t)$. Very important in the remainder of the course (using Laplace transforms)
- This result is proven in the appendix

Example 1: Solving a First Order ODE

- Calculate the response of a CT LTI system with impulse response:

$$h(t) = e^{-bt}u(t) \qquad b > 0$$

- to the input signal:

$$x(t) = e^{-at}u(t) \qquad a > 0$$

- Taking Fourier transforms of both signals:

$$H(j\omega) = \frac{1}{b + j\omega}, \qquad X(j\omega) = \frac{1}{a + j\omega}$$

- gives the overall frequency response:

$$Y(j\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

- to convert this to the time domain, express as **partial fractions**:

$$Y(j\omega) = \frac{1}{b - a}\left(\frac{1}{(a + j\omega)} - \frac{1}{(b + j\omega)}\right)$$

assume $b \neq a$

- Therefore, the CT system response is:

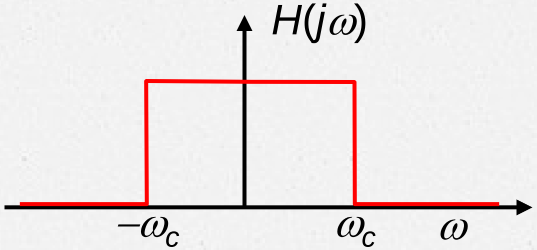
$$y(t) = \frac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Example 2: Design a Low Pass Filter

- Consider an ideal **low pass filter** in frequency domain:

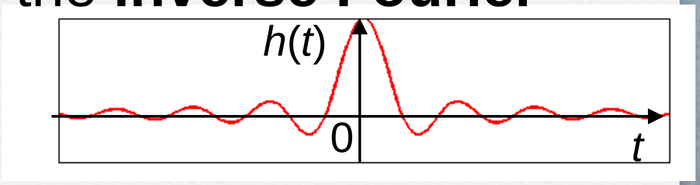
$$H(j\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & |\omega|>\omega_c\end{cases}$$

$$Y(j\omega)=\begin{cases}X(j\omega) & |\omega|<\omega_c\\ 0 & |\omega|>\omega_c\end{cases}$$



- The **filter's impulse response** is the **inverse Fourier transform**

$$h(t)=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega=\frac{\sin(\omega_c t)}{\pi t}$$



- which is an ideal low pass CT filter. However it is non-causal, so this cannot be manufactured exactly & the time-domain oscillations may be undesirable
- We need to approximate this filter with a causal system such as 1st order LTI system impulse response {$h(t)$, $H(j\omega)$}:

$$a^{-1}\frac{\partial y(t)}{\partial t}+y(t)=x(t),\qquad e^{-at}u(t)\overset{F}{\longleftrightarrow}\frac{1}{a+j\omega}$$

- The Fourier transform is widely used for designing **filters**. You can design systems with reject high frequency noise and just retain the low frequency components. This is natural to describe in the **frequency domain**.

- Important **properties** of the Fourier transform are:
- 1. **Linearity** and **time shifts**
- 2. **Differentiation**
- 3. **Convolution**

$$ax(t)+by(t)\overset{F}{\leftrightarrow}aX(j\omega)+bY(j\omega)$$

$$\frac{dx(t)}{dt}\overset{F}{\leftrightarrow}j\omega X(j\omega)$$

$$y(t)=h(t)*x(t)\overset{F}{\leftrightarrow}Y(j\omega)=H(j\omega)X(j\omega)$$

- Some operations are **simplified** in the frequency domain, but there are a number of signals for which the Fourier transform does not exist – this leads naturally onto **Laplace transforms**. Similar properties hold for Laplace transforms & the Laplace transform is widely used in engineering analysis.

Proof of Convolution Property

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Taking Fourier transforms gives:

$$Y(j\omega) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt$$

- Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right)d\tau$$

- By the time shift property, the bracketed term is $e^{-j\omega\tau}H(j\omega)$, so

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau$$

$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= H(j\omega)X(j\omega)$$