Lecture 5

Topics Covered

Mathematical Modelling

Chapter 2 mathematical models of systems

2.1 Introduction

2.1.1 Why？

  1) Easy to discuss the full possible types of the control systems —only in terms of the system's "mathematical characteristics".

  2) The basis of analyzing or designing the control systems.

2.1.2 What is？

  Mathematical models of systems — the mathematical relation-ships between the system's variables.

2.1.3 How get？

 1) theoretical approaches

 2) experimental approaches

 3) discrimination learning

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2.1.4 types

   1) Differential equations

   2) Transfer function

   3) Block diagram、signal flow graph

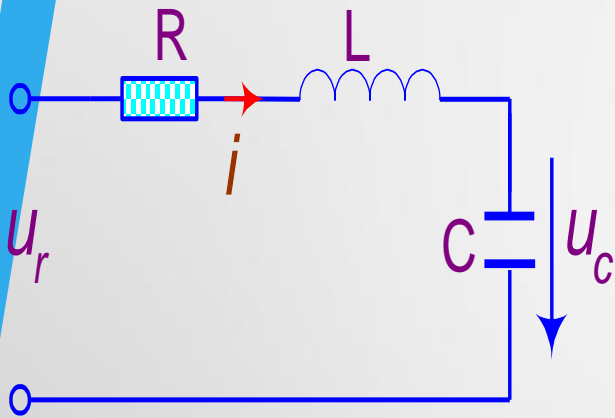   **4) State variables**

2.2 The input-output description of the physical systems —differential equations

The input-output description—description of the mathematical relationship between the output variable and the input variable of physical systems.

2.2.1 Examples

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Example 2.1 : A passive circuit

$define\colon\ input \to u_r \quad output \to u_{c\,\circ}$
$we\ have$:

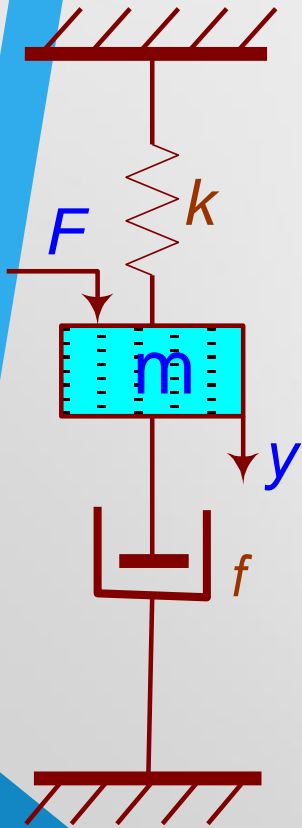$$Ri + L\frac{di}{dt} + u_c = u_r \quad i = C\frac{du_c}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2 u_c}{dt^2} + RC\frac{du_c}{dt} + u_c = u_r$$

$$make\colon\ RC = T1 \quad \frac{L}{R} = T2 \ \Rightarrow\ T_1 T_2\frac{d^2 u_c}{dt^2} + T_1\frac{du_c}{dt} + u_c = u_r$$

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Example 2.2 :   A mechanism

Define: *input* $\rightarrow F$,   *output* $\rightarrow y$.   *We have:*

$$F - ky - f\frac{dy}{dt} = m\frac{d^2y}{d^2t}$$

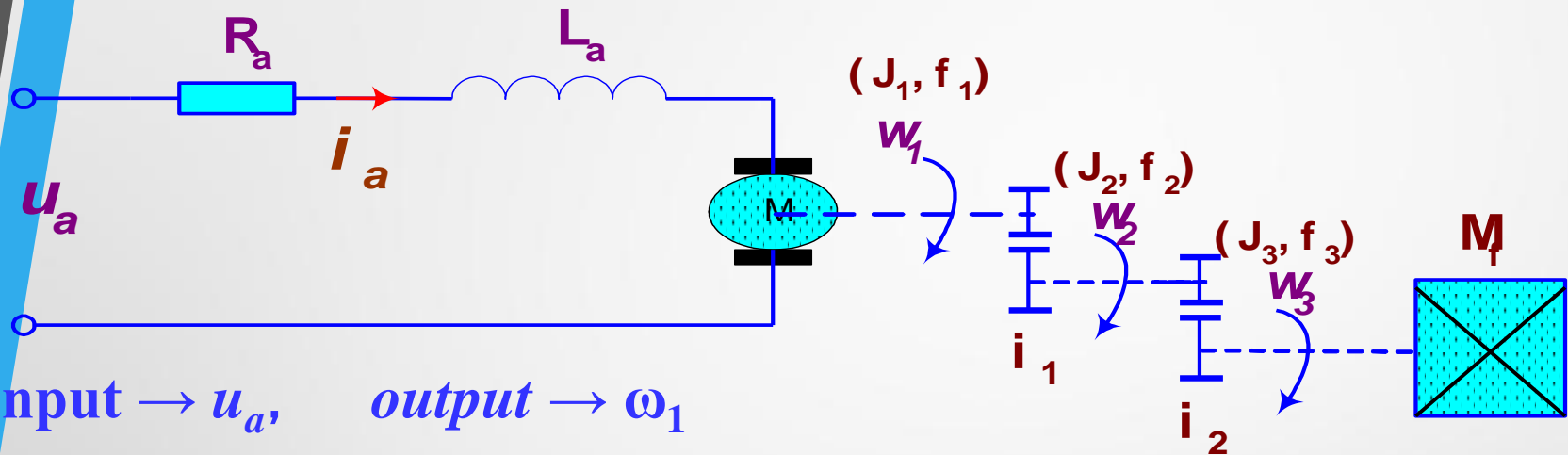$$\Downarrow$$

$$m\frac{d^2y}{dt^2} + f\frac{dy}{dt} + ky = F$$

If we make : $\dfrac{f}{k} = T_1,$ $\dfrac{m}{f} = T_2$

we have : $T_1T_2\dfrac{d^2y}{dt^2} + T_1\dfrac{dy}{dt} + y = \dfrac{1}{k}F$

Compare with example 2.1: $u_c \rightarrow y,\ u_r \rightarrow F$---*analogous systems*

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Example 2.4 : A DC motor

Input $\rightarrow u_a$,   output $\rightarrow \omega_1$

$$L_a \frac{di_a}{dt} + R_a i_a + E_a = u_a \ldots (1)$$

$$M = C_m i_a \ldots\ldots\ldots\ldots\ldots (2)$$

$$E_a = C_e \omega_1 \ldots\ldots\ldots\ldots\ldots (3)$$

$$\overline{M} - \overline{M} = \overline{J} \frac{d\omega_1}{dt} + \overline{f} \omega_1 \ldots (4)$$

$(4) \rightarrow (2) \rightarrow (1)$ and $(3) \rightarrow (1)$:

$$\frac{L_a \overline{J}}{C_e C_m} \overset{\bullet\bullet}{\omega_1} + \left(\frac{L_a \overline{f}}{C_e C_m} + \frac{R_a \overline{J}}{C_e C_m}\right) \overset{\bullet}{\omega_1} + \left(\frac{R_a \overline{f}}{C_e C_m} + 1\right) \omega_1$$

$$= \frac{1}{C_e} u_a - \frac{L_a}{C_e C_m} \overset{\bullet}{\overline{M}} - \frac{R_a}{C_e C_m} \overline{M}$$

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$$\overline{J} = J_1 + \frac{J_2}{i_1^2} + \frac{J_3}{i_1^2 i_2^2} \ ......equivalent \ \ moment \ \ of \ \ inertia$$

$$here: \ \ \overline{f} = f_1 + \frac{f_2}{i_1^2} + \frac{f_3}{i_1^2 i_2^2} \ ......equivalent \ friction \ coefficient$$

$$\overline{M} = \frac{M_f}{i_1 i_2} \ \ \ \ \ \ \ ....equivalent \ torque$$

$$(can \ be \ derived \ from: \omega_1 = i_1 \omega_2 = i_1 i_2 \omega_3)$$

$$Make: \ T_e = \frac{L_a}{R_a} \ ....electric \text{-} magnetic \ time \text{-} constant$$

$$T_m = \frac{R_a \overline{J}}{C_e C_m} \ ....mechanical \text{-} electric \ time \text{-} constant$$

$$T_f = \frac{R_a \overline{f}}{C_e C_m} \ ....friction \text{-} electric \ time \text{-} constant$$

the differential equation description of the DC motor is:

$$T_e T_m \overset{\bullet\bullet}{\omega_1} + (T_e T_f + T_m)\overset{\bullet}{\omega_1} + (T_f + 1)\omega_1$$

$$= \frac{1}{C_e} u_a - \frac{1}{\overline{J}}(T_e T_m \overset{\bullet}{\overline{M}} + T_m \overline{M})$$

Assume the motor idle: $M_f = 0$, **and neglect the friction:** $f = 0$, **we have:**
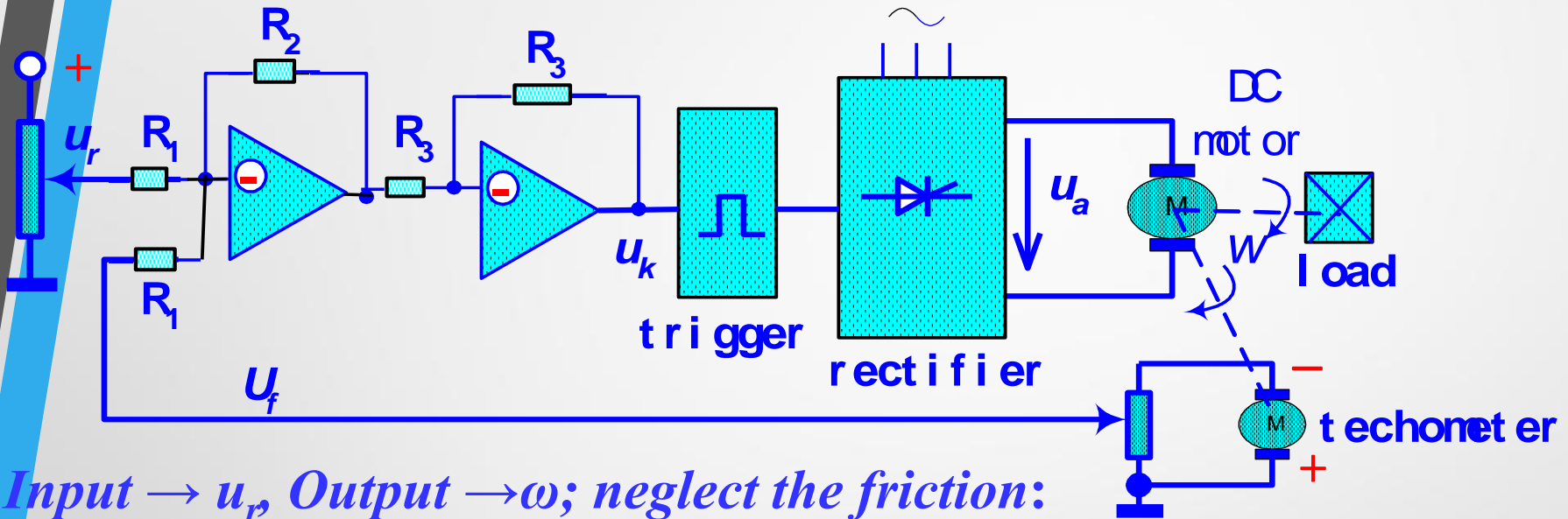
$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e} u_a$$

Compare with example 2.1 and example 2.2:

$$u_c \Leftrightarrow y \Leftrightarrow \omega; \quad u_r \Leftrightarrow F \Leftrightarrow u_a \text{ ----} Analogous\ systems$$

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Example 2.5 : A DC-Motor control system

Input → u_r, Output → ω; neglect the friction:

$$u_k = \frac{R_2}{R_1}(u_r - u_f) = k_1(u_r - u_f)...(1)$$

$$u_f = \alpha\,\omega.....................(2) \qquad\qquad u_a = k_2 u_k.......................(3)$$

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e}u_a - \frac{1}{J}(T_e T_m \dot{\overline{M}} + T_m \overline{M}).......(4)$$

（2） → （1） → （3） → （4）， **we have** :

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + (1 + k_1 k_2 \alpha \frac{1}{C_e})\omega = k_1 k_2 \frac{1}{C_e} u_r - \frac{T_m}{J}(T_e \dot{\overline{M}} + \overline{M})$$

2.2.2 steps to obtain the input-output description (differential equation) of control systems

1) **Identify the output and input variables of the control systems.**

2) **Write the differential equations of each system's component in terms of the physical laws of the components.**
   * **necessary assumption and neglect.**
   * **proper approximation.**

3) **dispel the intermediate(across) variables to get the input-output description which only contains the output and input variables.**

4) Formalize the input-output equation to be the "standard" form:
Input variable —— on the right of the input-output equation .
Output variable —— on the left of the input-output equation.
Writing the polynomial—according to the falling-power order.

2.2.3 General form of the input-output equation of the linear control systems

——A nth-order differential equation:

Suppose: $input \rightarrow r$, $output \rightarrow y$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y^{(1)} + a_n y$$

$$= b_0 r^{(m)} + b_1 r^{(m-1)} + b_2 r^{(m-2)} + \cdots + b_{m-1} r^{(1)} + b_m r ....n \geq m$$

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2.4 Transfer function

Another form of the input-output(external) description of control systems, different from the differential equations.

2.4.1 definition

Transfer function: *The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable* with all initial condition assumed to be zero and for the linear systems, that is:

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$$G(s) = \frac{C(s)}{R(s)}$$

C(s) —— Laplace transform of the output variable
R(s) —— Laplace transform of the input variable
G(s) —— transfer function

Notes:

* Only for the linear and stationary(constant parameter) systems.
* Zero initial conditions.
* Dependent on the configuration and coefficients of the systems, independent on the input and output variables.

2.4.2 How to obtain the transfer function of a system

1) If the impulse response *g(t)* is known

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We have: $\quad G(s) = L\big[g(t)\big]$

Because: $\quad G(s) = \dfrac{C(s)}{R(s)}, \quad \text{if } r(t) = \delta(t) \Rightarrow R(s) = 1$

Then: $\quad G(s) = C(s) = L\big[g(t)\big]$

Example 2.8 : $\quad g(t) = 5 - 3e^{-2t} \Rightarrow G(s) = \dfrac{5}{s} - \dfrac{3}{s+2} = \dfrac{2(s+5)}{s(s+2)}$

2) If the output response *c(t)* and the input *r(t)* are known

We have: $\quad G(s) = \dfrac{L\big[c(t)\big]}{L\big[r(t)\big]}$

Example 2.9: $r(t) = 1(t) \Rightarrow R(s) = \dfrac{1}{s} \dots\dots Unit\,step\,function$

$$c(t) = 1 - e^{-3t} \Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+3} = \frac{3}{s(s+3)}$$

$$\dots\dots Unit\,step\,response$$

Then: $$G(s) = \frac{C(s)}{R(s)} = \frac{3/s(s+3)}{1/s} = \frac{3}{s+3}$$

3) If the input-output differential equation is known

- **Assume: zero initial conditions;**
- **Make: Laplace transform of the differential equation;**
- **Deduce: *G(s)=C(s)/R(s).***

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Example 2.10:

$$2\ddot{c}(t) + 3\dot{c}(t) + 4c(t) = 5\dot{r}(t) + 6r(t)$$

$$\Downarrow$$
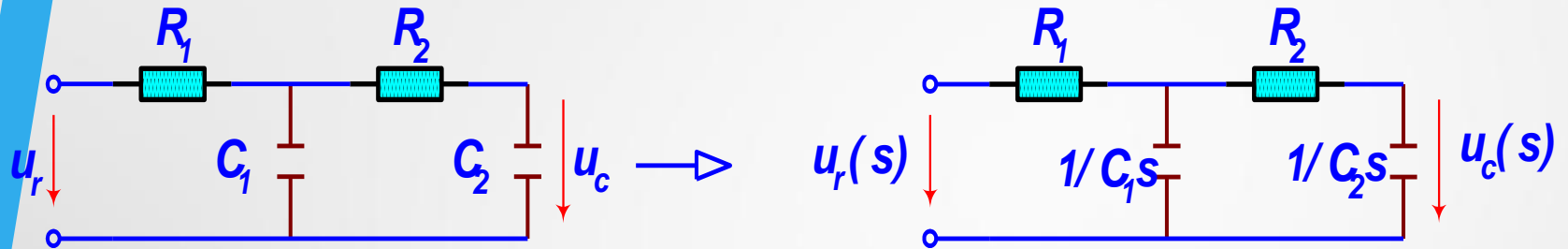
$$2s^2 C(s) + 3sC(s) + 4C(s) = 5sR(s) + 6R(s)$$

$$\Downarrow$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{5s + 6}{2s^2 + 3s + 4}$$

4) For a circuit

* **Transform a circuit into a operator circuit.**
* **Deduce the *C(s)/R(s)* in terms of the circuits theory.**

Example 2.11:   **For a electric circuit:**

$$U_c(s) = \frac{\dfrac{1}{sC_1} // (R_2 + \dfrac{1}{sC_2})}{R_1 + \dfrac{1}{sC_1} // (R_2 + \dfrac{1}{sC_2})} \cdot U_r(s) \cdot \frac{\dfrac{1}{sC_2}}{R_2 + \dfrac{1}{sC_2}}$$
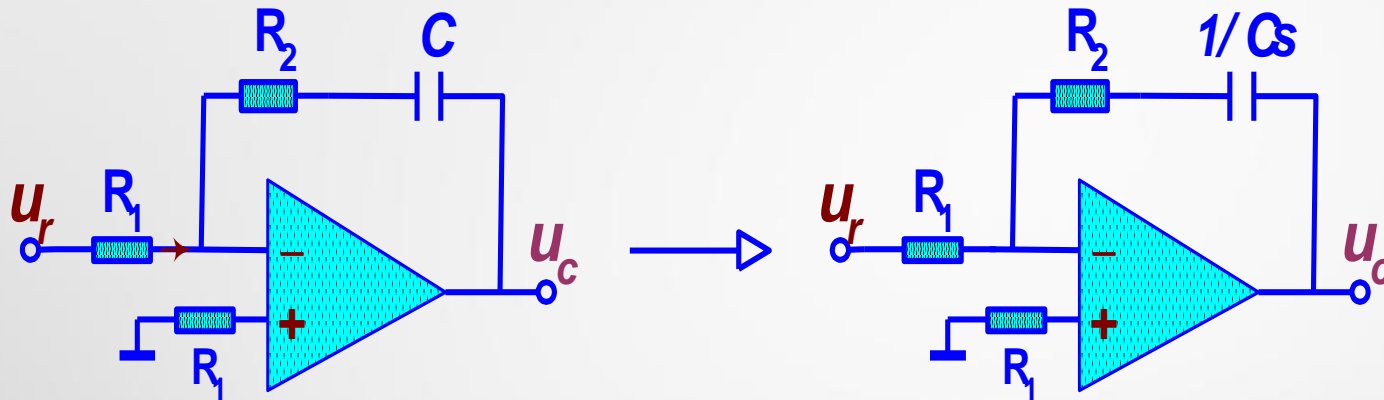
$$= \frac{1}{T_1 T_2 s^2 + (T_1 + T_2 + T_{12})s + 1} \cdot U_r(s)$$

$$G(s) = \frac{U_c(s)}{U_r(s)} = \frac{1}{T_1 T_2 s^2 + (T_1 + T_2 + T_{12})s + 1}$$

here : $T_1 = R_1 C_1;$ $T_2 = R_2 C_2;$ $T_{12} = R_1 C_2$

Example 2.12: For a op-amp circuit

$$G(s) = \frac{U_c(s)}{U_r(s)} = -\frac{R_2 + \dfrac{1}{sC}}{R_1} = -\frac{R_2Cs + 1}{R_1Cs}$$

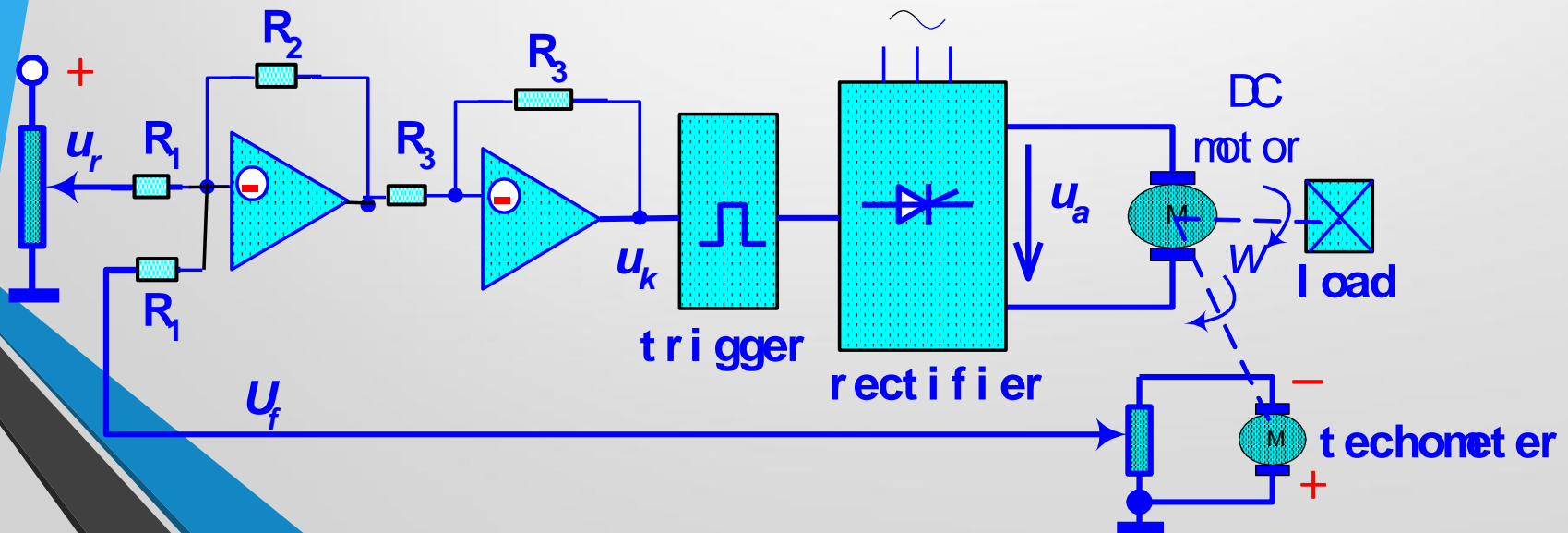$$= -k(1 + \frac{1}{\tau s})\ldots\ldots\ldots\ldots PI\text{-}Controller$$

$$here: \quad k = \frac{R_2}{R_1}; \quad \tau = R_2C \ldots\ldots Integral\ time\ constant$$

5) For a control system

- Write the differential equations of the control system;
- Make Laplace transformation, assume zero initial conditions, transform the differential equations into the relevant algebraic equations;
- Deduce: $G(s)=C(s)/R(s)$.

Example 2.13 the DC-Motor control system in Example 2.5

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In **Example 2.5,** we have written down the differential equations as:

$$u_k = \frac{R_2}{R_1}(u_r - u_f) = k_1(u_r - u_f)\dots\dots\dots\dots\dots\dots\dots(1)$$

$$u_f = \alpha\,\omega\dots\dots\dots\dots\dots(2) \qquad u_a = k_2 u_k \dots\dots\dots\dots\dots(3)$$

$$T_e T_m \frac{d^2\omega}{dt^2} + T_m \frac{d\omega}{dt} + \omega = \frac{1}{C_e}u_a - \frac{T_m}{J}(T_e\,\dot{\overline{M}} + \overline{M})\dots\dots(4)$$

Make Laplace transformation, we have:

$$U_k(s) = k_1[U_r(s) - U_f(s)]\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$U_f(s) = \alpha\Omega(s)\dots\dots\dots\dots(2) \qquad U_a(s) = k_2 U_k(s)\dots\dots\dots\dots(3)$$

$$(T_e T_m s^2 + T_m s + 1)\Omega(s) = \frac{1}{Ce}U_a(s) - \frac{T_e T_m s + T_m}{J}\overline{M}(s)\dots\dots(4)$$

$(2)\rightarrow(1)\rightarrow(3)\rightarrow(4)$, we have:

$$[T_e T_m s^2 + T_m s + (1 + k_1 k_2 \alpha \frac{1}{C_e})]\Omega(s) = k_1 k_2 \frac{1}{C_e} U_r(s) - \frac{T_e T_m s + T_m}{J} \overline{M}(s)$$

$$G(s) = \frac{\Omega(s)}{U_r(s)} = \frac{k_1 k_2 \ {1}/{C_e}}{T_e T_m s^2 + T_m s + (1 + k_1 k_2 \alpha \frac{1}{C_e})}$$

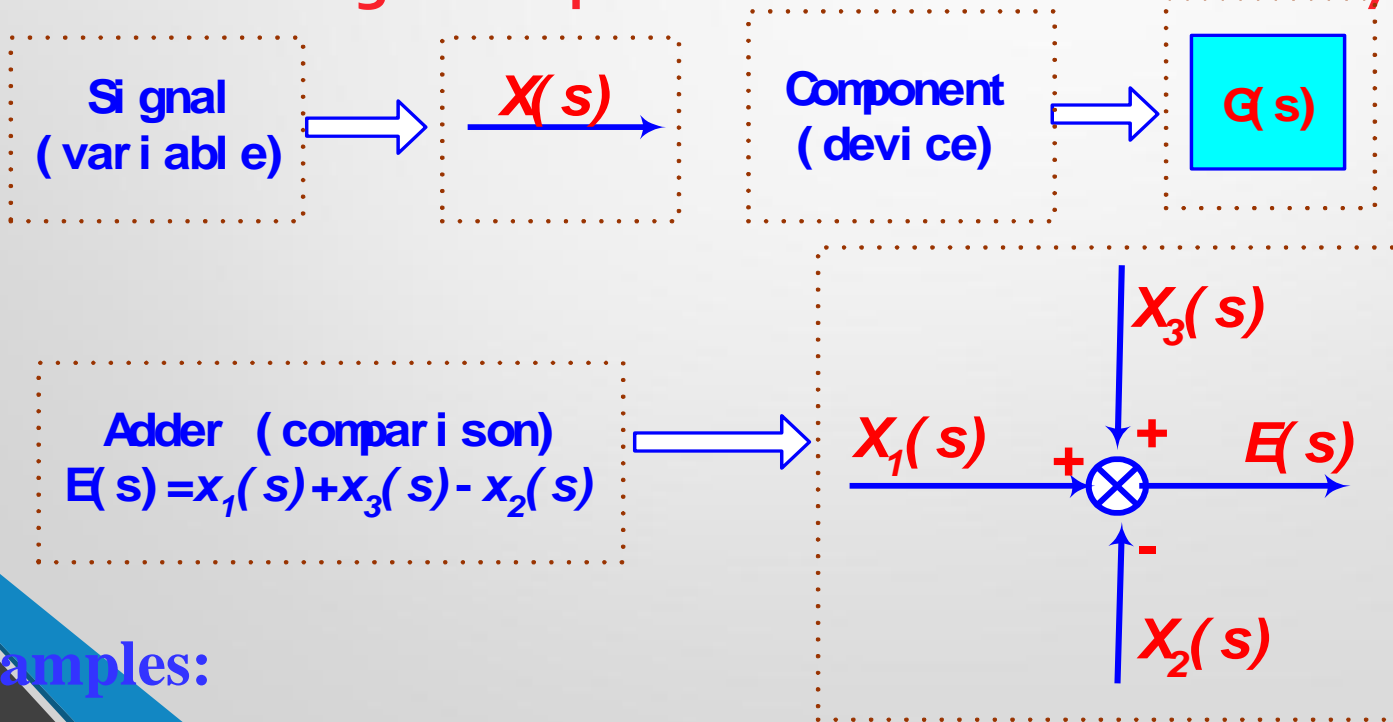$$here: \ T_e = \frac{L_a}{R_a} ....electric - magnetic \ \ time \ \text{-} \ constant$$

$$T_m = \frac{R_a J}{C_e C_m} ......mechanical - electric \ \ time \ \text{-} \ constant$$

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2.6   block diagram models (dynamic)

Portray the control systems by the block diagram models more intuitively than the transfer function or differential equation models

.2.6.1 Block diagram representation of the control systems

| Signal (variable) | $\Rightarrow$ | $X(s)$ $\longrightarrow$ | Component (device) | $\Rightarrow$ | $G(s)$ |

Adder (comparison)
$E(s) = x_1(s) + x_3(s) - x_2(s)$

$\Rightarrow$

$X_3(s)$

$X_1(s)$ $+$ $+$ $E(s)$
$\otimes$

$-$

$X_2(s)$

Examples:

Example 2.14   **For the DC motor in Example 2.4**

In Example 2.4, we have written down the differential equations as:

$$L_a \frac{di_a}{dt} + R_a i_a + E_a = u_a \ ....(1) \qquad M = C_m i_a ....(2)$$

$$E_a = C_e \omega ....(3) \qquad M - \overline{M} = \overline{J} \frac{d\omega}{dt} + \overline{f} \omega .....(4)$$

Make Laplace transformation, we have:

$$L_a s I_a(s) + R_a I_a(s) + E_a(s) = U_a(s) \ \Rightarrow \ I_a(s) = \frac{U_a(s) - E_a(s)}{L_a s + R_a} ....(5)$$

$$M(s) = C_m I_a(s)..(6)$$

$$E_a(s) = C_e \Omega(s)..(7)$$

$$M(s) - \overline{M}(s) = \overline{J} s \Omega(s) + \overline{f} \Omega(s) \ \Rightarrow \ \Omega(s) = \frac{1}{\overline{J} s + \overline{f}}[M(s) - \overline{M}(s)]......(8)$$
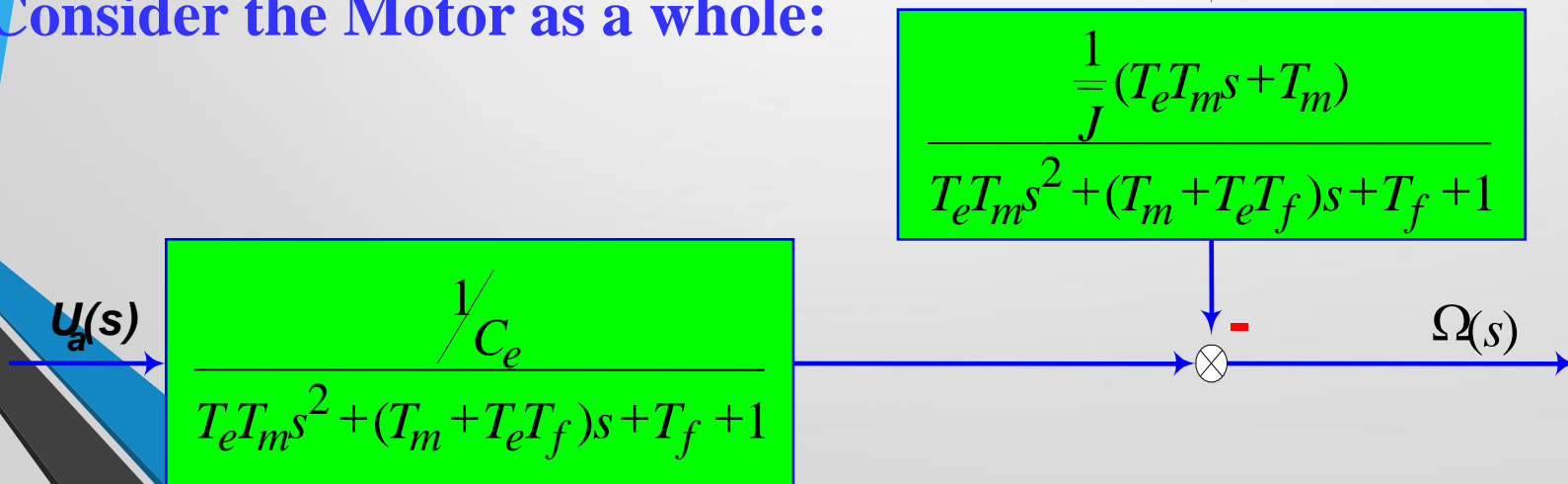
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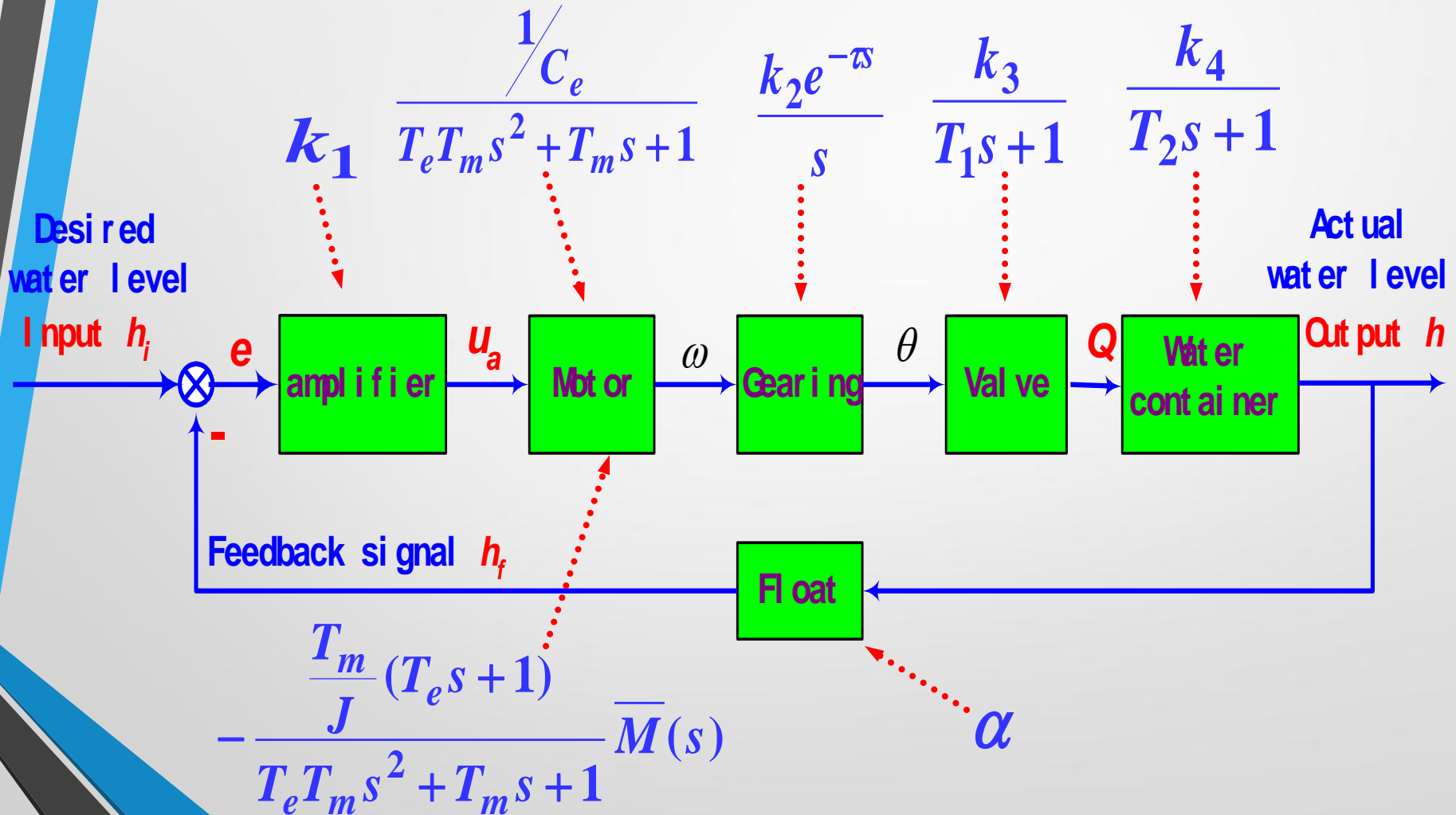
Draw block diagram in terms of the equations (5)~(8):
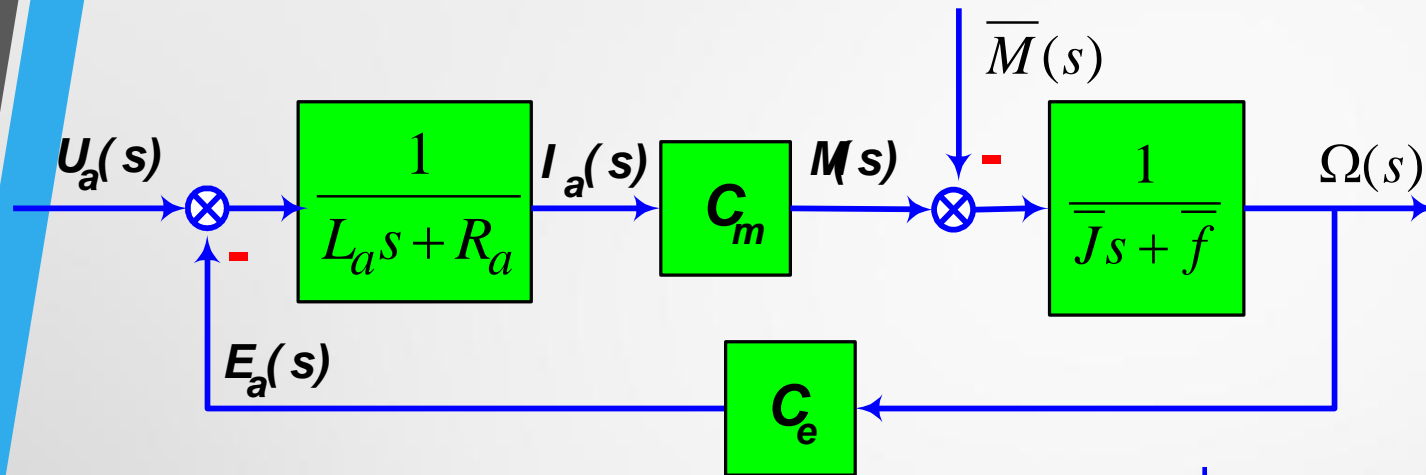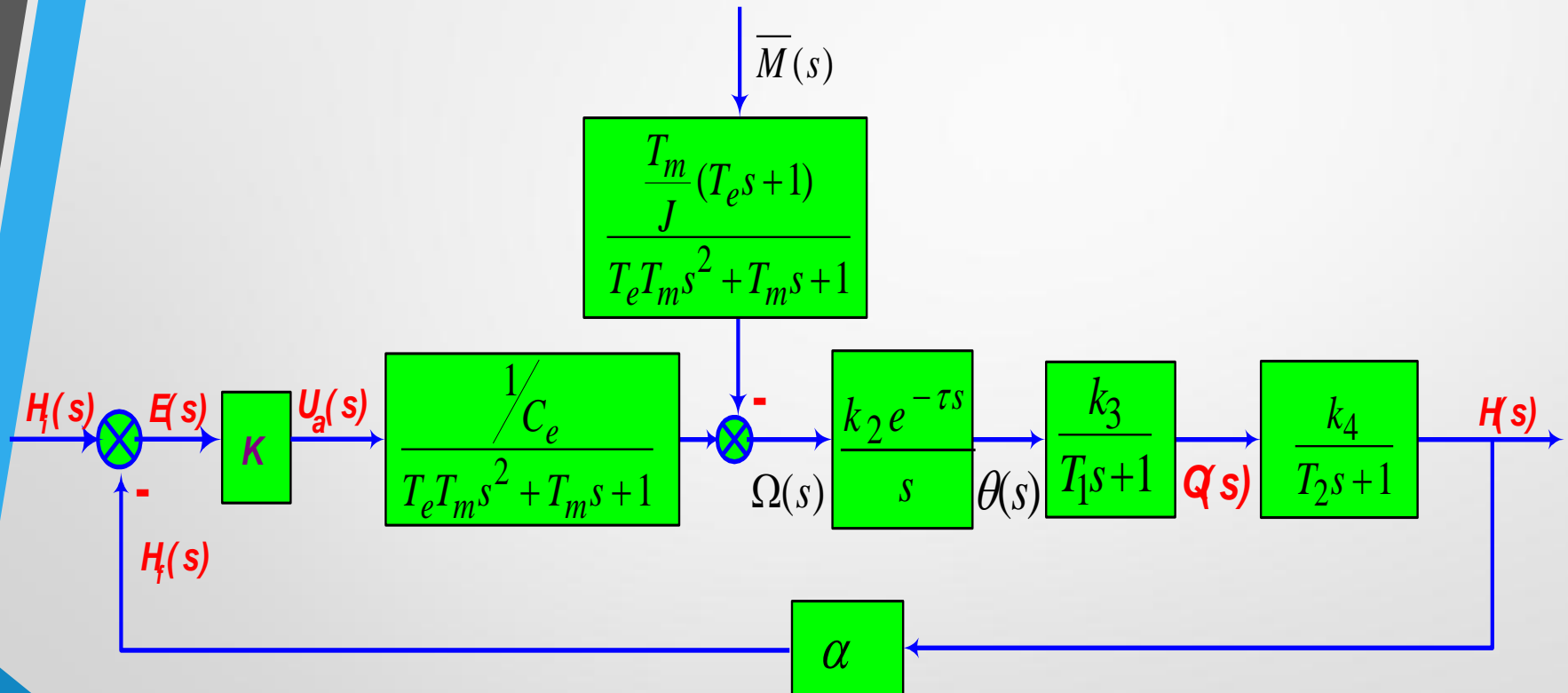

Consider the Motor as a whole:

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Example 2.15   **The water level control system in Fig 1.8:**

$$k_1 \qquad \dfrac{1\!\big/\!C_e}{T_e T_m s^2 + T_m s + 1} \qquad \dfrac{k_2 e^{-\tau s}}{s} \qquad \dfrac{k_3}{T_1 s + 1} \qquad \dfrac{k_4}{T_2 s + 1}$$

Desired
water level

Input h_i

e amplifier u_a Motor ω Gearing θ Valve Q Water container

Actual
water level

Output h

-

Feedback signal h_f

Float

$$-\dfrac{\dfrac{T_m}{J}(T_e s + 1)}{T_e T_m s^2 + T_m s + 1} \overline{M}(s)$$

α

The block diagram model is:

Example 2.16 **The DC motor control system in Fig 1.9**

The block diagram model is:

$$\overline{M}(s)$$

$$\frac{\dfrac{T_m}{J}(T_e s + 1)}{T_e T_m s^2 + T_m s + 1}$$

$U_r(s)$ $E(s)$ $k_p(1+\dfrac{1}{T_I s})$ $U_k(s)$ $k_D e^{-\tau s}$ $U_a(s)$ $\dfrac{1/C_e}{T_e T_m s^2 + T_m s + 1}$ **-** $\Omega(s)$ $\dfrac{1}{s}$ $\theta(s)$

-

$U_f(s)$

$\dfrac{\alpha}{Ts+1}$