SYSTEM SIMULATION AND MODELLING

LECTURE 1

Section C

Topic Covered: Useful Statistical Models, Discrete Distributions, Continuous Distributions, Poisson Process, Empirical Distributions

DISCRETE DISTRIBUTION

- Discrete distribution is shown in Table 5.2.

Table 5.2

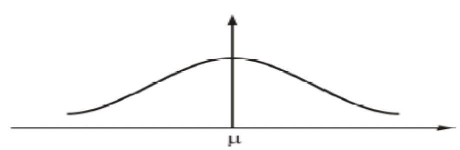
S.No.	*Discrete distribution*
1.	**Bernoulli distribution:** The n Bernoulli trials are called a Bernoulli process if the trials are independent, each trial has only two possible outcomes (success or failure), and the probability of a success remain constant from trial to trial thus $p(x_1, x_2, \ldots x_n) = p_1(x_1) \cdot p_2(x_2) \ldots p_n(x_n)$ ...(5.7) and $p_j(x_j) = p(x_j) = \begin{cases} p & x_j = 1, j = 1, 2, \ldots, n \\ 1 - p = q & x_j = 0, j = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$ ...(5.8) for one trial, the distribution given in equation (5.8) is called the Bernoulli distribution. • Mean of X_j are calculated as follows: $E[X_j] = 0 \cdot q + 1 \cdot p = p$ ...(5.9) • Variance of X_j are calculated as follow: $\sigma_x^2 = [(0^2 \cdot q) + (1^2 \cdot p)] - p^2 = p(1 - p)$ ...(5.10)

2.	**Binomial Distribution:** The random variable X that denotes the number of successes in n Bernoulli trials has a binomial distribution given by $p(x)$, when $$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, \ldots n \\ 0, & \text{otherwise} \end{cases} \quad \ldots(5.11)$$ where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ • Mean $E[X] = p + p + \ldots + p = np$ ...(5.12) • Variance $\sigma_x^2 = pq + pq + \ldots + pq = npq$ ...(5.13)
3.	**Poisson Distribution:** The Poisson probability mass function is given by, $$p(x) = \begin{cases} \frac{e^{-\alpha}\alpha^x}{x!}, & x = 0, 1, \ldots \\ 0, & \text{otherwise} \end{cases}$$ Parameter α = average number of events occuring within the interval where $\alpha > 0$. ...(5.14) • Mean $E[X] = \alpha$ ...(5.15) • Variance $\sigma_x^2 = \alpha$ ...(5.16) **Note:** Mean and variance are equal in Poisson distribution. The cumulative distribution function CDF is given by $$F_X(x) = \sum_{i=0}^{x} \frac{e^{-\alpha}\alpha^i}{i!}$$
4.	**Geometric distribution:** The geometric distribution is related to a sequence of Bernoulli trials. Random variable X is defined as the number of trials required to achieve the first success. The distribution of X is given by $$p(x) = \begin{cases} q^{x-1} p, & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \ldots(5.17)$$ where $q = 1 - p$ • Mean $E[X] = \frac{1}{p}$ ...(5.18) • Variance $\sigma_x^2 = \frac{1-p}{p^2}$ ...(5.19)
5.	**Negative Binomial Distributions:** The negative binomial distribution is the distribution of the number of trials until the K^{th} success for $K = 1, 2, \ldots$. If Y has a negative binomial distribution with parameters p and K, then the distribution of Y is given by $$p(y) = \begin{cases} \binom{y-1}{K-1} q^{y-K} p^K = y = K, K+1, K+2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \ldots(5.20)$$ • Mean $E[Y] = K/p$ ...(5.21) • Variance $\sigma_x^2 = \frac{Kq}{p^2}$ ...(5.22)

CONTINUOUS DISTRIBUTION

- Continuous distribution is shown in Table 5.3.

Table 5.3

S.No.	*Continuous distribution*
1.	**Exponential Distribution:** A random variable X is said to be exponentially distributed with parameter λ if its pdf is given by $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{...(5.24)}$$ where λ is a rate: arrivals per hour or service per minute. The exponential distribution has been used to model interarrival times when arrivals are completely random and to model service time that are highly variable. • Mean $E[X] = \frac{1}{\lambda}$ ...(5.25) • Variance $\sigma_x^2 = \frac{1}{\lambda^2}$ ...(5.26) **Note:** The mean and standard deviation are equal in exponential distribution.
2.	**Normal Distribution:** A random variable X with mean $-\infty < \mu < \infty$ and variance σ_x^2 has a normal distribution if it has the pdf $$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2\right], \; -\infty < x < \infty \quad \text{...(5.27)}$$ PDF of the normal distribution is shown in Fig. 5.4.  **Fig. 5.4** PDF of the normal distribution.

3. **Uniform Distribution:** A random variable X is uniformly distributed on the interval (a, b) if its PDF is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases} \quad \text{...(5.28)}$$

- Mean $E[X] = \frac{a+b}{2}$ ...(5.29)
- Variance $\sigma_x^2 = \frac{(b-a)^2}{12}$ ...(5.30)

4. **Weibull Distribution:** The random variable X has a Weibull distribution if its PDF has the form

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right], & x \ge \nu \\ 0 & \text{otherwise} \end{cases} \quad \text{...(5.31)}$$

where scale parameter $\alpha > 0$, shape parameter $\beta > 0$ and location parameter $-\infty < \nu < \infty$

- Mean $E[X] = \nu + \alpha\Gamma\left(\frac{1}{\beta}+1\right)$ ...(5.32)
- Variance $\sigma_x^2 = \alpha^2\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2\right]$ ...(5.33)

5. **Lognormal Distribution:** A random variable X has a lognormal distribution if its PDF is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma_x^2}\right], & x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{...(5.34)}$$

where $\sigma_x^2 > 0$.

- Mean $E[X] = e^{\mu+\sigma^2/2}$ ...(5.35)
- Variance $\sigma_x^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$ ...(5.36)

Note: The parameter μ and σ_x^2 are not the mean and variance of the lognormal.

If the mean and variance of the lognormal to be μ_Z and σ_Z^2 respectively, then the parameters μ and σ_x^2 are given by

$$\mu = \ln\left(\frac{\mu_Z^2}{\sqrt{\mu_Z^2 + \sigma_Z^2}}\right) \quad \text{...(5.37)}$$

$$\sigma_x^2 = \ln\left(\frac{\mu_Z^2 + \sigma_Z^2}{\mu_Z^2}\right) \quad \text{...(5.38)}$$

POISSON DISTRIBUTION

The Poisson distribution was first derived in 1837 by the French mathematician Simeon Denis Poisson whose main work was on the mathematical theory of electricity and magnetism.

The Poisson probability mass function is given by

$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^{x}}{x!}, & x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(5.52)$$

where $\alpha > 0$.

- The cumulative distribution function is given by

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\alpha} \alpha^{i}}{i!} \qquad ...(5.53)$$