Course Name: Analysis and Design of Algorithms

Topics to be covered

- *NP*-Completeness
- Travelling salesman problem
- P and NP

NP-Completeness

- So far we've seen a lot of good news!
  - Such-and-such a problem can be solved quickly (i.e., in close to linear time, or at least a time that is some small polynomial function of the input size)
- NP-completeness is a form of bad news!
  - Evidence that many important problems can not be solved quickly.
- NP-complete problems really come up all the time!

Why should we care?

- Knowing that they are hard lets you stop beating your head against a wall trying to solve them…
  - **Use a heuristic:** come up with a method for solving a reasonable fraction of the common cases.
  - **Solve approximately:** come up with a solution that you can prove that is close to right.
  - **Use an exponential time solution:** if you really have to solve the problem exactly and stop worrying about finding a better solution.

Optimization & Decision Problems

- **Decision problems**
  - Given an input and a question regarding a problem, determine if the answer is yes or no
- **Optimization problems**
  - Find a solution with the "best" value
- Optimization problems can be cast as decision problems that are easier to study
  - *E.g.:* Shortest path: G = unweighted directed graph
    - Find a path between u and v that uses the fewest edges
    - *Does a path exist from u to v consisting of at most k edges?*

Algorithmic vs Problem Complexity

- The ***algorithmic complexity*** of a computation is some measure of how *difficult* is to perform the computation (i.e., specific to an algorithm)
- The **complexity of a computational *problem*** or *task* is the complexity of the algorithm with the **lowest** order of growth of complexity for solving that problem or performing that task.
  - *e.g.* the problem of searching an ordered list has *at most lgn* time complexity.
- **Computational Complexity**: deals with classifying problems by how hard they are.

Class of "P" Problems

- **Class P** consists of (decision) problems that are solvable in polynomial time
- Polynomial-time algorithms
  - Worst-case running time is $O(n^k)$, for some constant k
- Examples of polynomial time:
  - $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
- Examples of non-polynomial time:
  - $O(2^n)$, $O(n^n)$, $O(n!)$

Tractable/Intractable Problems

- Problems in P are also called **tractable**
- Problems **not** in P are **intractable or unsolvable**
  - Can be solved in reasonable time only for small inputs
  - Or, can not be solved at all
- Are non-polynomial algorithms always worst than polynomial algorithms?

  - $n^{1,000,000}$ is *technically* tractable, but really impossible - $n^{\log \log \log n}$ is *technically* intractable, but easy

Example of Unsolvable Problem

- Turing discovered in the 1930's that there are problems **unsolvable** by *any* algorithm.
- The most famous of them is the ***halting problem***
  - Given an arbitrary algorithm and its input, will that algorithm eventually halt, or will it continue forever in an "*infinite loop*?"

Examples of Intractable Problems

Hamiltonian Paths

Optimization Problem: Given a graph, find a path that passes through every vertex exactly once

Decision Problem: Does a given graph have a Hamiltonian Path ?

Traveling Salesman

Optimization Problem: Find a minimum weight Hamiltonian Path

Decision Problem: Given a graph and an integer k, is there a Hamiltonian Path with a total weight at most k ?

Intractable Problems

- Can be classified in various categories based on their degree of difficulty, e.g.,
  - NP
  - NP-complete
  - NP-hard
- Let’s define NP algorithms and NP problems …

Nondeterministic and NP Algorithms

Nondeterministic algorithm = two stage procedure:

1) Nondeterministic ("guessing") stage:

   generate randomly an arbitrary string that can be thought of as a candidate solution ("certificate")

2) Deterministic ("verification") stage:

   take the certificate and the instance to the problem and returns YES if the certificate represents a solution

NP algorithms (Nondeterministic polynomial)

verification stage is polynomial

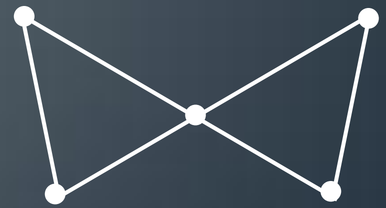
Class of “NP” Problems

- **Class NP** consists of problems that could be solved by NP algorithms
  - i.e., verifiable in polynomial time
- If we were given a “certificate” of a solution, we could verify that the certificate is correct in time polynomial to the size of the input
- <u>Warning:</u> NP does **not** mean “non-polynomial”

E.g.: Hamiltonian Cycle

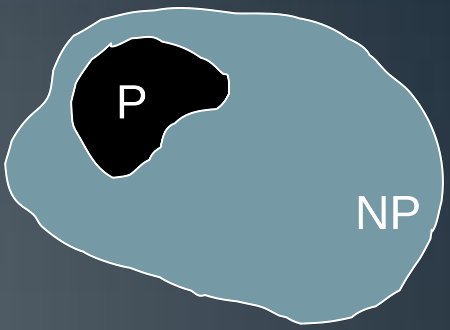
- **Given:** a directed graph G = (V, E), determine a simple cycle that contains each vertex in V
  - Each vertex can only be visited once
- **Certificate**:
  - Sequence: $\langle v_1, v_2, v_3, \ldots, v_{|V|} \rangle$

hamiltonian



not hamiltonian

Is P = NP?

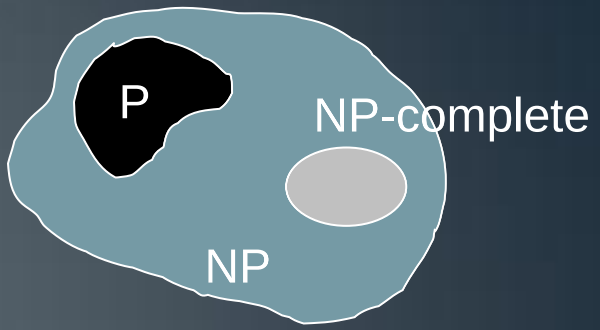


- Any problem in P is also in NP:

$$P \subseteq NP$$

- The big (and **open question**) is whether NP $\subseteq$ P or P = NP
  - i.e., if it is always easy to check a solution, should it also be easy to find a solution?
- Most computer scientists believe that this is false but we do not have a proof …

NP-Completeness (informally)

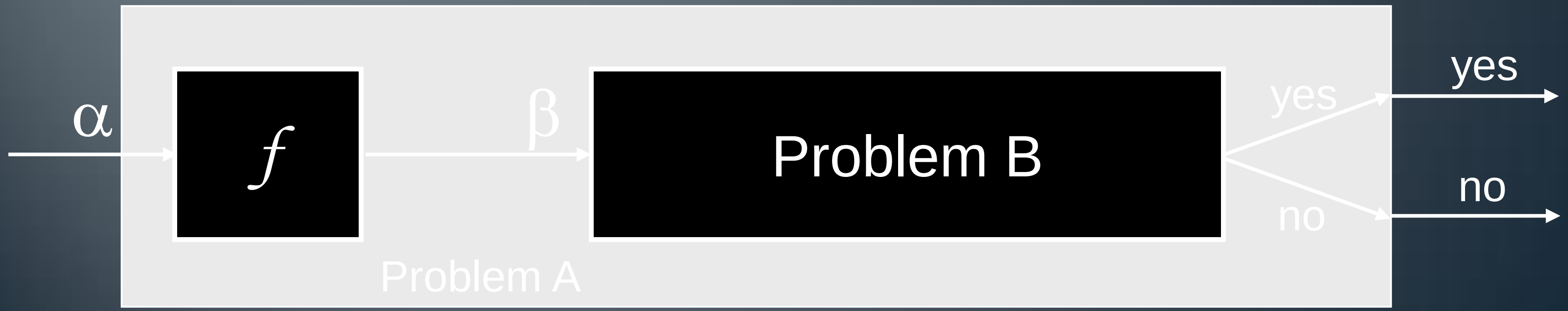


- **NP-complete** problems are defined as the hardest problems in NP
- Most practical problems turn out to be either P or NP-complete.
- Study NP-complete problems …

Reductions

- Reduction is a way of saying that one problem is **"easier"** than another.
- We say that problem A is easier than problem B, (i.e., we write **"A ≤ B"**)

  if we can solve A using the algorithm that solves B.
- **Idea:** transform the inputs of A to inputs of B

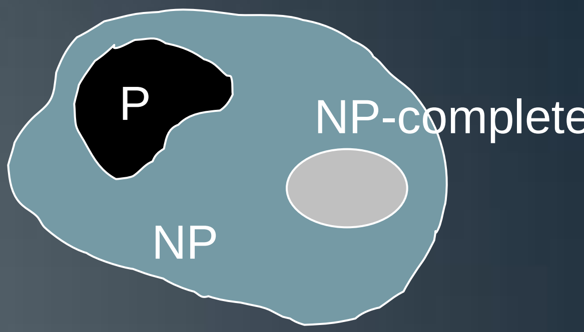


Polynomial Reductions

- Given two problems A, B, we say that A is polynomially **reducible** to B ($A \leq_p B$) if:

1. There exists a function f that converts the input of A to inputs of B in polynomial time
2. A(i) = YES $\Leftrightarrow$ B(f(i)) = YES

NP-Completeness (formally)



- A problem B is **NP-complete** if:

  (1) $B \in$ **NP**

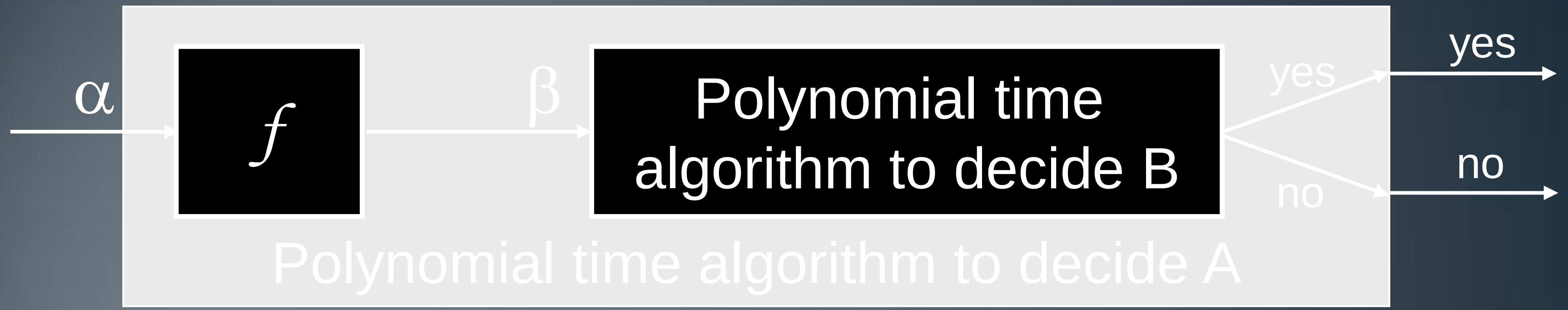
  (2) $A \leq_p B$ for all $A \in$ **NP**

- If B satisfies only property (2) we say that B is **NP-hard**
- No polynomial time algorithm has been discovered for an **NP-Complete** problem
- No one has ever proven that no polynomial time algorithm can exist for any **NP-Complete** problem

Implications of Reduction

- If A $\leq_p$ B and B $\in$ P, then A $\in$ P
- if A $\leq_p$ B and A $\notin$ P, then B $\notin$ P

Proving Polynomial Time

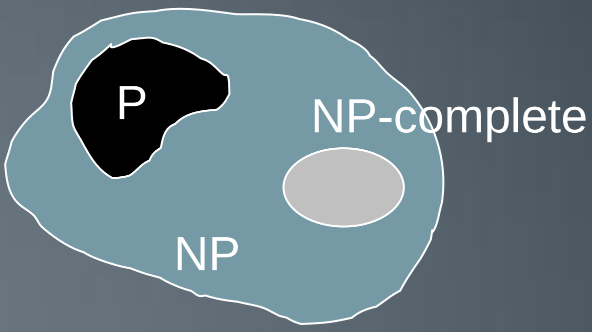


1. Use a **polynomial time** reduction algorithm to transform A into B
2. Run a known **polynomial time** algorithm for B
3. Use the answer for B as the answer for A

Proving NP-Completeness In Practice

- Prove that the problem B is in NP
  - A randomly generated string can be checked in polynomial time to determine if it represents a solution
- Show that **one known** NP-Complete problem can be transformed to B in polynomial time
  - No need to check that **all** NP-Complete problems are reducible to B

Revisit "Is P = NP?"



Theorem: If any NP-Complete problem can be solved in polynomial time $\Rightarrow$ then P = NP.

P & NP-Complete Problems

- **Shortest simple path**
  - Given a graph G = (V, E) find a **shortest** path from a source to all other vertices
  - Polynomial solution: O(VE)
- **Longest simple path**
  - Given a graph G = (V, E) find a **longest** path from a source to all other vertices
  - NP-complete

P & NP-Complete Problems

- **Euler tour**
  - G = (V, E) a connected, directed graph find a cycle that traverses <u>each edge</u> of G exactly once (may visit a vertex multiple times)
  - <u>Polynomial solution O(E)</u>
- **Hamiltonian cycle**
  - G = (V, E) a connected, directed graph find a cycle that visits <u>each vertex</u> of G exactly once
  - <u>NP-complete</u>

Satisfiability Problem (SAT)

- **Satisfiability problem:** given a logical expression Φ, find an assignment of values (F, T) to variables x_i that causes Φ to evaluate to T

$$\Phi = x_1 \vee \neg x_2 \wedge x_3 \vee \neg x_4$$

- SAT was the first problem shown to be NP-complete!

CFN Satisfiability

- CFN is a special case of SAT
- Φ is in "Conjuctive Normal Form" (CNF)
  - "AND" of expressions (i.e., clauses)
  - Each clause contains only "OR"s of the variables and their complements

E.g.: $\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$

clauses

3-CNF Satisfiability

A subcase of CNF problem:

- Contains three clauses

- *E.g.:*

  $\Phi = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

- **3-CNF** is NP-Complete

- Interestingly enough, **2-CNF** is in P!

Clique

Clique Problem:

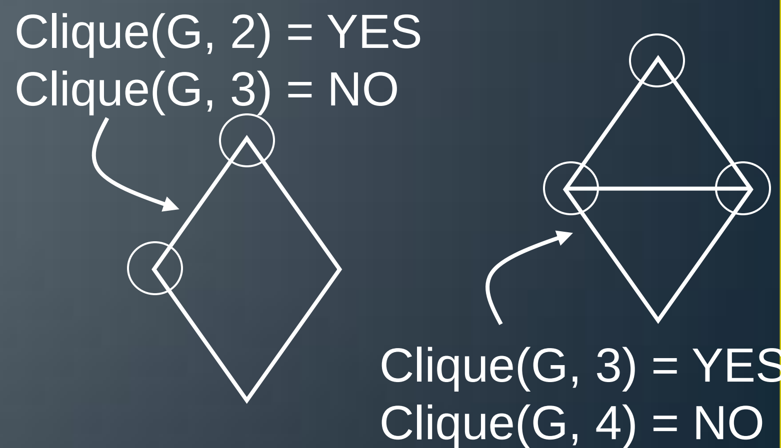
- Undirected graph G = (V, E)
- **Clique:** a subset of vertices in V all connected to each other by edges in E (i.e., forming a complete graph)
- **Size of a clique:** number of vertices it contains

Optimization problem:

- Find a clique of maximum size

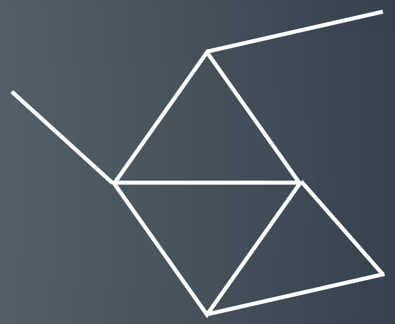
Decision problem:

- Does G have a clique of size k?



Clique Verifier

- **Given**: an undirected graph G = (V, E)
- **Problem**: Does G have a clique of size k?
- **Certificate**:
  - A set of k nodes
- **Verifier**:
  - Verify that for all pairs of vertices in this set there exists an edge in E



3-CNF $\leq_p$ Clique

- **Idea:**
  - Construct a graph G such that Φ is satisfiable only if G has a clique of size k

NP-naming convention

- **NP-complete -** means problems that are 'complete' in NP, i.e. the most difficult to solve in NP
- **NP-hard** - stands for 'at least' as hard as NP (but not necessarily **in** NP);
- **NP-easy** - stands for 'at most' as hard as NP (but not necessarily **in** NP);
- **NP-equivalent** - means equally difficult as NP, (but not necessarily **in** NP);

Examples NP-complete and NP-hard problems

Hamiltonian Paths

NP-complete

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NP-hard

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