Course Name: Analysis and Design of Algorithms

Topics to be covered

- Decision and Optimization Problems
- Polynomial-Time Reducibility
- NP-Hardness and NP-Completeness
- Examples: TSP, Circuit-SAT, Knapsack
- Polynomial-Time Approximation Schemes

Decision and Optimization Problems

- Decision Problem: computational problem with intended output of “yes” or “no”, 1 or 0
- Optimization Problem: computational problem where we try to maximize or minimize some value
- Introduce parameter k and ask if the optimal value for the problem is a most or at least k. Turn optimization into decision

Complexity Class P

- Deterministic in nature
- Solved by conventional computers in polynomial time
  - $O(1)$ Constant
  - $O(\log n)$ Sub-linear
  - $O(n)$ Linear
  - $O(n \log n)$ Nearly Linear
  - $O(n^2)$ Quadratic
- Polynomial upper and lower bounds

Complexity Class NP

- Non-deterministic part as well
- choose(b): choose a bit in a non-deterministic way and assign to b
- If someone tells us the solution to a problem, we can verify it in polynomial time
- Two Properties: non-deterministic method to generate possible solutions, deterministic method to verify in polynomial time that the solution is correct.

Relation of P and NP

- P is a subset of NP
- “P = NP”?
- Language L is in NP, complement of L is in co-NP
- co-NP ≠ NP
- P ≠ co-NP

Polynomial-Time Reducibility

- Language L is polynomial-time reducible to language M if there is a function computable in polynomial time that takes an input x of L and transforms it to an input f(x) of M, such that x is a member of L if and only if f(x) is a member of M.
- Shorthand, $L^{poly}M$ means L is polynomial-time reducible to M

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NP-Hard and NP-Complete

- Language M is NP-hard if every other language L in NP is polynomial-time reducible to M
- For every L that is a member of NP, $L^{poly}M$
- If language M is NP-hard and also in the class of NP itself, then M is NP-complete

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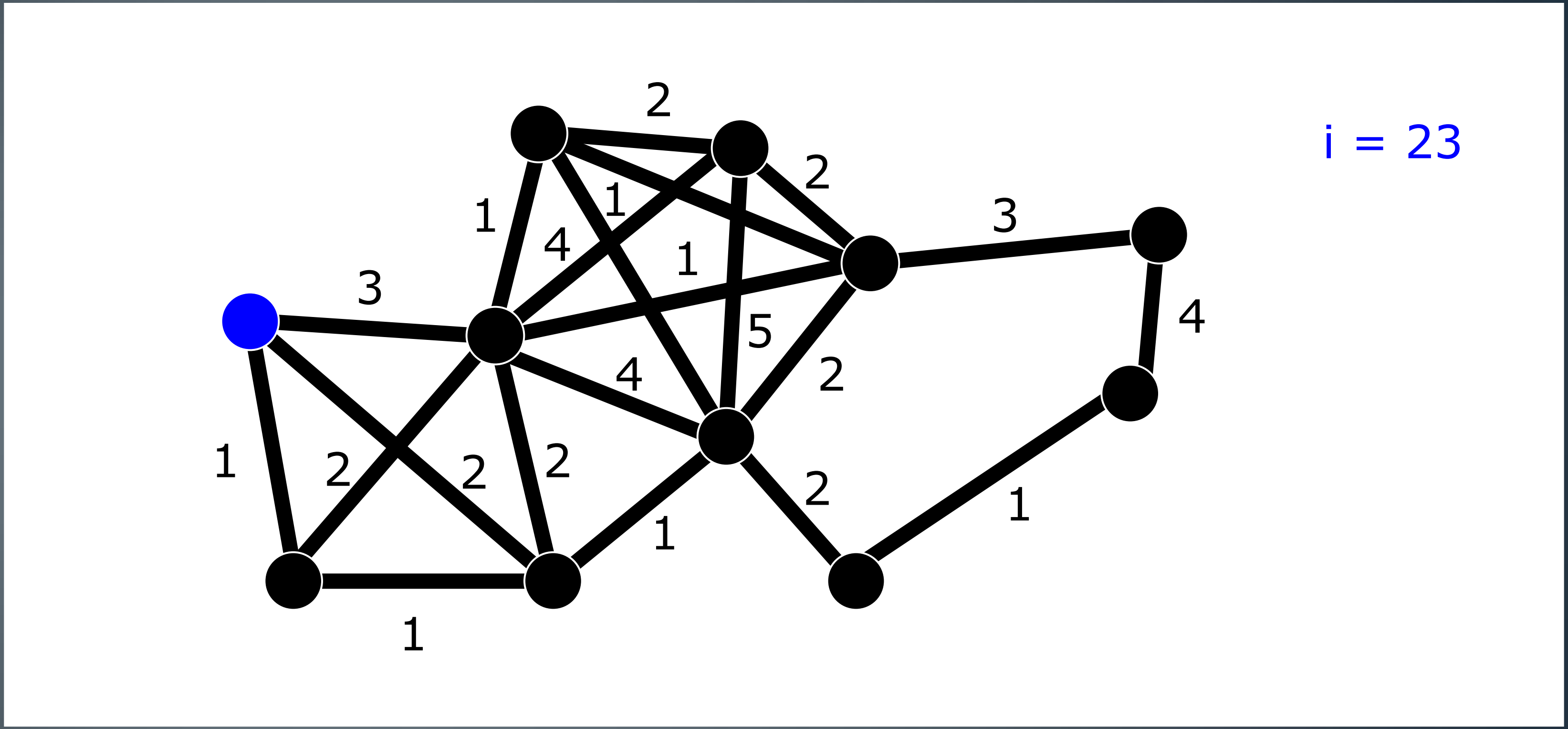
NP-Hard and NP-Complete

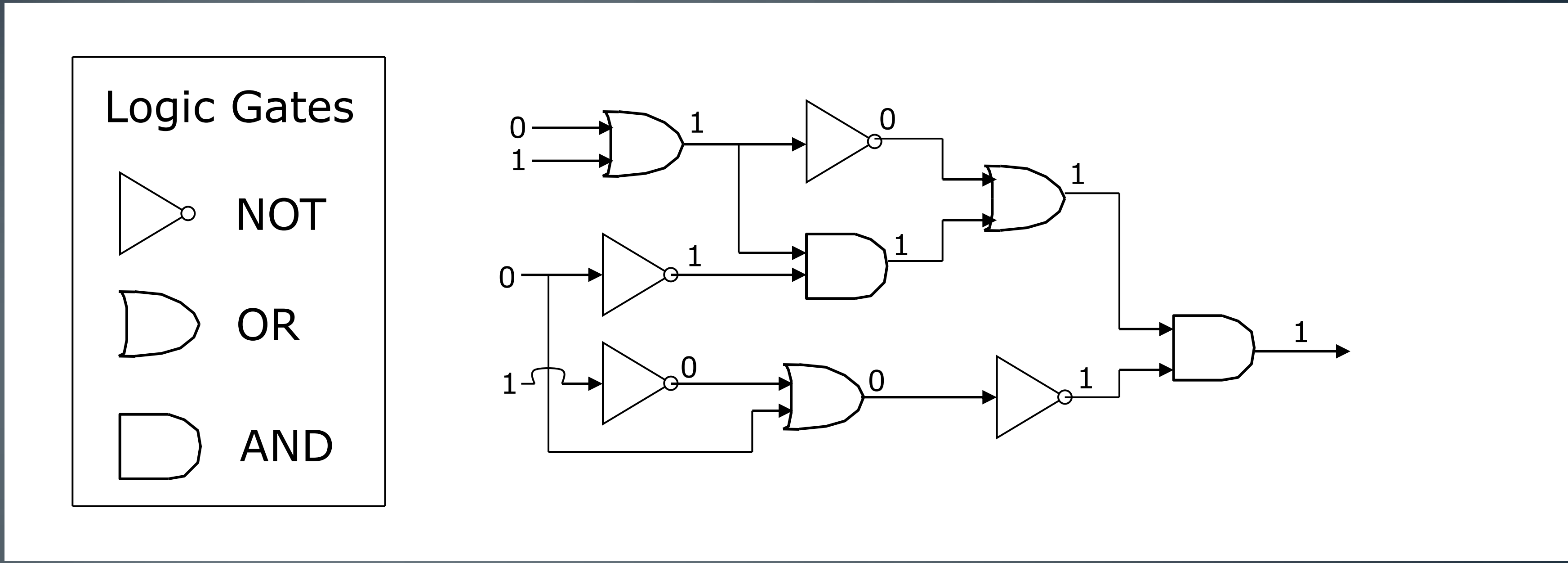
- Restriction: A known NP-complete problem M is actually just a special case of L
- Local replacement: reduce a known NP-complete problem M to L by dividing instances of M and L into “basic units” then showing each unit of M can be converted to a unit of L
- Component design: reduce a known NP-complete problem M to L by building components for an instance of L that enforce important structural functions for instances of M.

TSP



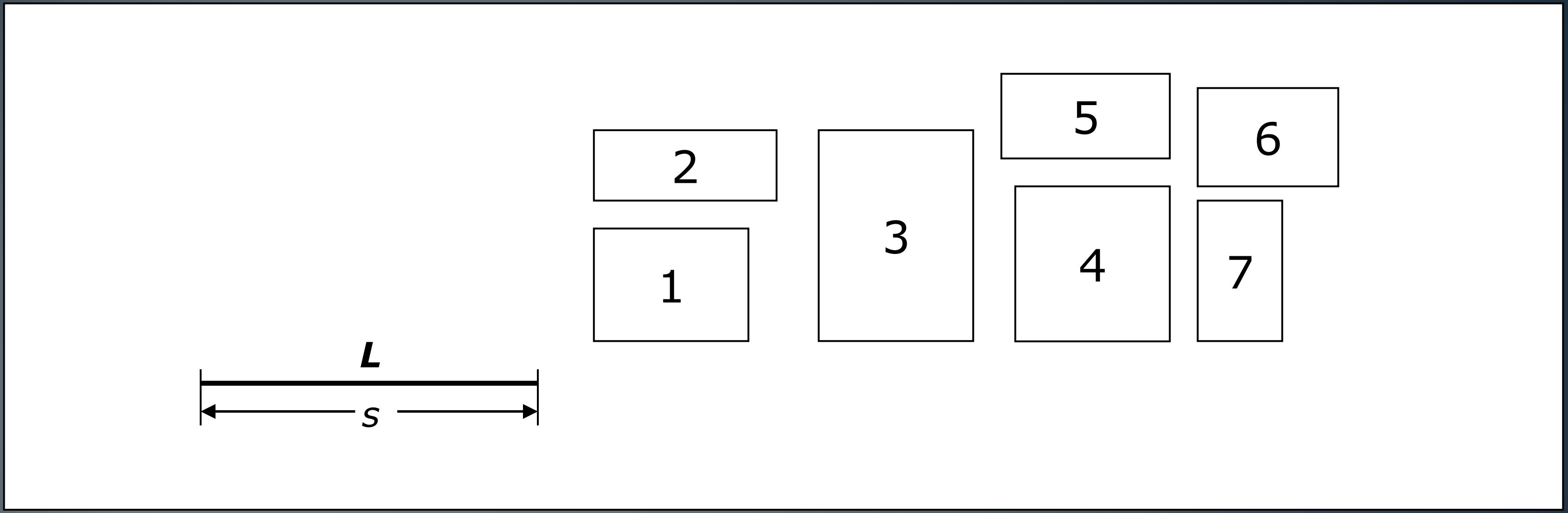
- For each two cities, an integer cost is given to travel from one of the two cities to the other. The salesperson wants to make a minimum cost circuit visiting each city exactly once.

Circuit-SAT



- Take a Boolean circuit with a single output node and ask whether there is an assignment of values to the circuit's inputs so that the output is "1"

Knapsack



- Given *s* and *w* can we translate a subset of rectangles to have their bottom edges on L so that the total area of the rectangles touching L is at least *w*?

PTAS

- Polynomial-Time Approximation Schemes
- Much faster, but not guaranteed to find the best solution
- Come as close to the optimum value as possible in a reasonable amount of time
- Take advantage of rescalability property of some hard problems

Backtracking

- Effective for decision problems
- Systematically traverse through possible paths to locate solutions or dead ends
- At the end of the path, algorithm is left with (x, y) pair. x is remaining subproblem, y is set of choices made to get to x
- Initially (x, Ø) passed to algorithm

Algorithm Backtrack(x):

Input: A problem instance x for a hard problem

Output: A solution for x or "no solution" if none exists

```
F ← {(x, Ø)}.
while F ≠ Ø do
    select from F the most "promising" configuration (x, y)
    expand (x, y) by making a small set of additional choices
    let (x_1, y_1), …, (x_k, y_k) be the set of new configurations.
    for each new configuration (x_i, y_i) do
        perform a simple consistency check on (x_i, y_i)
        if the check returns "solution found" then
            return the solution derived from (x_i, y_i)
        if the check returns "dead end" then
            discard the configuration (x_i, y_i)
        else
            F ← F U {(x_i, y_i)}.
return "no solution"
```

Branch-and-Bound

- Effective for optimization problems
- Extended Backtracking Algorithm
- Instead of stopping once a single solution is found, continue searching until the best solution is found
- Has a scoring mechanism to choose most promising configuration in each iteration

Algorithm Branch-and-Bound(x):

Input: A problem instance *x* for a hard optimization problem

Output: A solution for *x* or "no solution" if none exists

```
F ← {(x, Ø)}.
b ← {(+∞, Ø)}.
while F ≠ Ø do
    select from F the most "promising" configuration (x, y)
    expand (x, y), yielding new configurations (x_1, y_1), …, (x_k, y_k)
    for each new configuration (x_i, y_i) do
        perform a simple consistency check on (x_i, y_i)
        if the check returns "solution found" then
            if the cost c of the solution for (x_i, y_i) beats b then
                b ← (c, (x_i, y_i))
            else
                discard the configuration (x_i, y_i)
        if the check returns "dead end" then
            discard the configuration (x_i, y_i)
        else
            if lb(x_i, y_i) is less than the cost of b then
                F ← F U {(x_i, y_i)}.
            else
                discard the configuration (x_i, y_i)
return b
```

Summary

- Decision and Optimization Problems
- P and NP
- Polynomial-Time Reducibility
- NP-Hardness and NP-Completeness
- TSP, Circuit-SAT, Knapsack
- PTAS
- Backtracking/Branch-and-Bound

References

- A.K. Dewdney, The New Turning Omnibus, pp. 276-281, 357-362, Henry Holt and Company, 2001.
- Goodrich & Tamassia, Algorithm Design, pp. 592-637, John Wiley & Sons, Inc., 2002.