# Course Name: Analysis and Design of Algorithms

# Topics to be covered

- Dynamic Programming
  - Optimal Binary Search Tree

# Optimal Binary Search Trees

- Problem
  - Given sequence $K = k_1 < k_2 <\cdots < k_n$ of $n$ sorted keys, with a search probability $p_i$ for each key $k_i$.
  - Want to build a binary search tree (BST) with minimum expected search cost.
  - Actual cost = # of items examined.
  - For key $k_i$, cost = $\mathrm{depth}_T(k_i)+1$, where $\mathrm{depth}_T(k_i)$ = depth of $k_i$ in BST $T$ .

# Expected Search Cost

$$E[\text{search cost in } T]$$

$$= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i$$

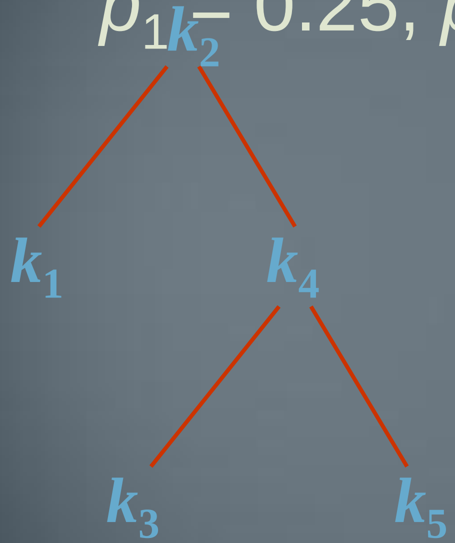
$$= \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i + \sum_{i=1}^{n} p_i$$

Sum of probabilities is 1.

$$= 1 + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i \qquad (15.16)$$

# Example

- Consider 5 keys with these search probabilities: $p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.

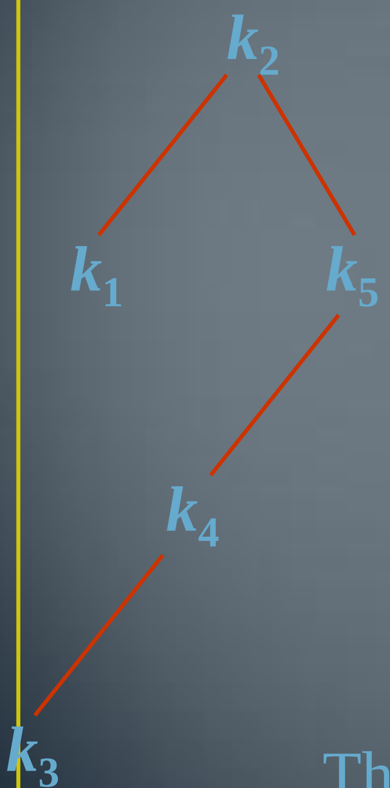



| $i$ | $\text{depth}_T(k_i)$ | $\text{depth}_T(k_i) \cdot p_i$ |
|---|---|---|
| 1 | 1 | 0.25 |
| 2 | 0 | 0 |
| 3 | 2 | 0.1 |
| 4 | 1 | 0.2 |
| 5 | 2 | 0.6 |
| | | 1.15 |

Therefore, E[search cost] = 2.15.

# Example

- $p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.




| $i$ | $\text{depth}_T(k_i)$ | $\text{depth}_T(k_i) \cdot p_i$ |
|---|---|---|
| 1 | 1 | 0.25 |
| 2 | 0 | 0 |
| 3 | 3 | 0.15 |
| 4 | 2 | 0.4 |
| 5 | 1 | 0.3 |
| | | 1.10 |

Therefore, E[search cost] = 2.10.

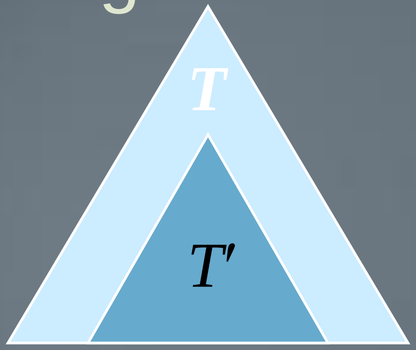
This tree turns out to be optimal for this set of keys.

# Example

- **Observations:**
  - Optimal BST may not have smallest height.
  - Optimal BST may not have highest-probability key at root.
- Build by exhaustive checking?
  - Construct each *n*-node BST.
  - For each,
    assign keys and compute expected search cost.
  - But there are $\Omega(4^n/n^{3/2})$ different BSTs with *n* nodes.

# Optimal Substructure

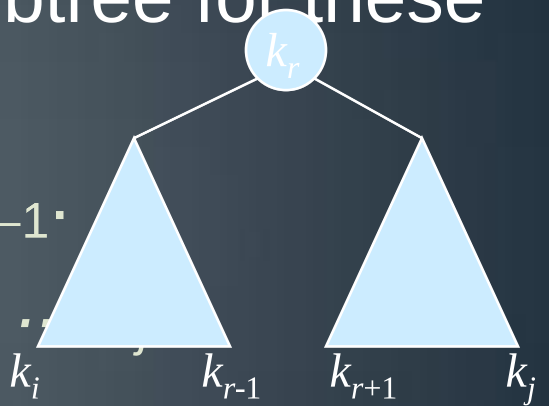
- Any subtree of a BST contains keys in a contiguous range $k_i, \ldots, k_j$ for some $1 \le i \le j \le n$.




- If $T$ is an optimal BST and
  $T$ contains subtree $T'$ with keys $k_i, \ldots, k_j$,
  then $T'$ must be an optimal BST for keys $k_i, \ldots, k_j$.
- **Proof:** Cut and paste.

# Optimal Substructure

- One of the keys in $k_i, \ldots, k_j$, say $k_r$, where $i \le r \le j$, must be the root of an optimal subtree for these keys.
- Left subtree of $k_r$ contains $k_i, \ldots, k_{r-1}$.
- Right subtree of $k_r$ contains $k_r$+1, …




- To find an optimal BST:
  - Examine all candidate roots $k_r$, for $i \le r \le j$
  - Determine all optimal BSTs containing $k_i, \ldots, k_{r-1}$ and containing $k_{r+1}, \ldots, k_j$

# Recursive Solution

- Find optimal BST for $k_i,\ldots,k_j$, where $i \geq 1$, $j \leq n$, $j \geq i-1$. When $j = i-1$, the tree is empty.
- Define $e[i, j\,]$ = expected search cost of optimal BST for $k_i,\ldots,k_j$.

- If $j = i-1$, then $e[i, j\,] = 0$.
- If $j \geq i$,
  - Select a root $k_r$, for some $i \leq r \leq j$ .
  - Recursively make an optimal BSTs
    - for $k_i,\ldots,k_{r-1}$ as the left subtree, and
    - for $k_{r+1},\ldots,k_j$ as the right subtree.

# Recursive Solution

- When the OPT subtree becomes a subtree of a node:
  - Depth of every node in OPT subtree goes up by 1.
  - Expected search cost increases by

$$w(i, j) = \sum_{l=i}^{j} p_l$$

from (15.16)

- If $k_r$ is the root of an optimal BST for $k_i,..,k_j$ :
  - $e[i, j\,] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$

    $= e[i, r-1] + e[r+1, j] + w(i, j)$. (because $w(i, j) = w(i, r-1) + p_r + w(r + 1, j)$)
- But, we don't know $k_r$. Hence,

$$e[i, j] = \begin{cases} 0 & \text{if } \ j = i - 1 \\ \min\limits_{i \le r \le j} \{e[i, r-1] + e[r+1, j] + w(i, j)\} & \text{if } i \le j \end{cases}$$

# Computing an Optimal Solution

For each subproblem $(i,j)$, store:

- expected search cost in a table $e[1 \ldots n+1\,,\, 0 \ldots n]$
  - Will use only entries $e[i, j\,]$, where $j \geq i-1$.
- $\mathrm{root}[i, j\,]$ = root of subtree with keys $k_i,\ldots,k_j$, for $1 \leq i \leq j \leq n$.
- $w[1..n+1, 0..n]$ = sum of probabilities
  - $w[i, i-1] = 0$ for $1 \leq i \leq n$.
  - $w[i, j\,] = w[i, j\text{-}1] + p_j$ for $1 \leq i \leq j \leq n$.

# Pseudo-code

```
OPTIMAL-BST(p, q, n)
for i ← 1 to n + 1
    do e[i, i– 1] ← 0
       w[i, i– 1] ← 0
for l ← 1 to n
    do for i ← 1 to n–l + 1
        do j ←i + l–1
           e[i, j ]←∞
           w[i, j ] ← w[i, j–1] + p_j
           for r ←i to j
               do t ← e[i, r–1] + e[r + 1, j ] + w[i, j ]
                  if t < e[i, j ]
                      then e[i, j ] ← t
                           root[i, j ] ←r
return e and root
```

**Consider all trees with *l* keys.**

**Fix the first key.**

**Fix the last key**

**Determine the root of the optimal (sub)tree**

Time: $O(n^3)$

# Elements of Dynamic Programming

- Optimal substructure
- Overlapping subproblems

# Optimal Substructure

- Show that a solution to a problem consists of making a choice, which leaves one or more subproblems to solve.
- Suppose that you are given this last choice that leads to an optimal solution.
- Given this choice, determine which subproblems arise and how to characterize the resulting space of subproblems.
- Show that the solutions to the subproblems used within the optimal solution must themselves be optimal. Usually use cut-and-paste.
- Need to ensure that a wide enough range of choices and subproblems are considered.

# Optimal Substructure

- Optimal substructure varies across problem domains:
  - 1. *How many subproblems* are used in an optimal solution.
  - 2. *How many choices* in determining which subproblem(s) to use.
- Informally, running time depends on (# of subproblems overall) × (# of choices).
- How many subproblems and choices do the examples considered contain?
- Dynamic programming uses optimal substructure **bottom up**.
  - *First* find optimal solutions to subproblems.
  - *Then* choose which to use in optimal solution to the problem.

# Optimal Substucture

- Does optimal substructure apply to all optimization problems? No.
- Applies to determining the shortest path but NOT the longest simple path of an unweighted directed graph.
- Why?
  - Shortest path has independent subproblems.
  - Solution to one subproblem does not affect solution to another subproblem of the same problem.
  - Subproblems are not independent in longest simple path.
    - Solution to one subproblem affects the solutions to other subproblems.
  - Example:

# Overlapping Subproblems

- The space of subproblems must be “small”.
- The total number of distinct subproblems is a polynomial in the input size.
  - A recursive algorithm is exponential because it solves the same problems repeatedly.
  - If divide-and-conquer is applicable, then each problem solved will be brand new.