Course Name: Analysis and Design of Algorithms

Topics to be covered

- Union and Find for Disjoint Data sets

Union-Find Problem

- Given a set {1, 2, …, n} of n elements
- Initially each element is in a different set
  - {1}, {2}, …, {n}
- An intermixed sequence of union and find operations is performed
- A union operation combines two sets into one
  - Each of the n elements is in exactly one set at any time
  - Can be proven by induction
- A find operation identifies the set that contains a particular element
- Application – Equivalence Class

Disjoint Sets

- Suppose we have N distinct items. We want to partition the items into a collection of sets such that:
  - each item is in a set
  - no item is in more than one set
- Examples
  - BU students according to majors, or
  - BU students according to GPA, or
  - Graph vertices according to connected components
- The resulting sets are said to be ***disjoint sets***.

Disjoint sets

- **Set** : a collection of (distinguishable) elements
- Two sets are **disjoint** if they have no common elements
- Disjoint-set data structure:
  - maintains a collection of disjoint sets
  - each set has a representative element
  - supported operations:
    - MakeSet(x)
    - Find(x)
    - Union(x,y)

Disjoint sets

- **MakeSet(x)**
  - Given object x, create a new set whose only element (and representative) is pointed to by x
- **Find(x)**
  - Given object x, return (a pointer to) the representative of the set containing x
  - Assumption: there is a pointer to each x so we never have to look for an element in the structure

Disjoint sets

- **Union(x,y)**
  - Given two elements x, y, merge the disjoint sets containing them.
  - The original sets are destroyed.
  - The new set has a new representative (usually one of the representatives of the original sets)
  - At most *n-1* Unions can be performed where *n* is the number of elements (why?)

Union-Find Algorithms

Disjoint set algorithms are sometimes called ***union-find*** algorithms.

Disjoint Set Example

Find the connected components of the undirected graph G=(V,E) (maximal subgraphs that are connected).

```
for (each vertex v in V)
    Makeset(v): put v in its own set
for (each edge (u,v) in E)
    if (find(u) ~= find(v))
        union(u,v)
```

Now we can find if two vertices x and y are in the same connected component by testing

```
find(x) == find(y)
```

Disjoint sets -- implementation

In the discussion that follows:

- *n* is the total number of elements (in all sets).
- *m* is the total number of operations performed

Disjoint Sets:Implementation #1

- **Using linked lists:**
  - The first element of the list is the representative
  - Each node contains:
    - an element
    - a pointer to the next node in the list
    - a pointer to the representative

Disjoint Sets: Implementation#1

- **Using linked lists:**
  - MakeSet(x)
    - Create a list with only one node, for x
    - Time O(1)
  - Find(x)
    - Return the pointer to the representative (assuming you are pointing at the x node)
    - Time O(1)

Disjoint Sets:Implementation#1

- **Using linked lists:**
  - Union(x,y)
    1 . Append y's list to x's list.
    2 . Pick x as a representative
    3 . Update y's "representative" pointers
    - A sequence of m operations may take O(m^2) time
    - Improvement: let each representative keep track of the length of its list and always *append the shorter list to the longer one.*
      - Now, a sequence of m operations takes O(m+nlgn) time (why?)

Disjoint Sets:Implementation#1 An Improvement

- Let each representative keep track of the length of its list and always *append the shorter list to the longer one.*
- Theorem: Any sequence of m operations takes $O(m+n \log n)$ time.

Disjoint Sets:Implementation#2

- **Using arrays:**
  - Keep an array of size *n*
  - Cell *i* of the array holds the representative of the set containing *i*.
  - Similar to lists, simpler to implement.

A Tight Bound

- O(n + u log u + f), where u and f are, respectively, the number of union and find operations in the sequence of requests

- Can we do better?

Up-Trees

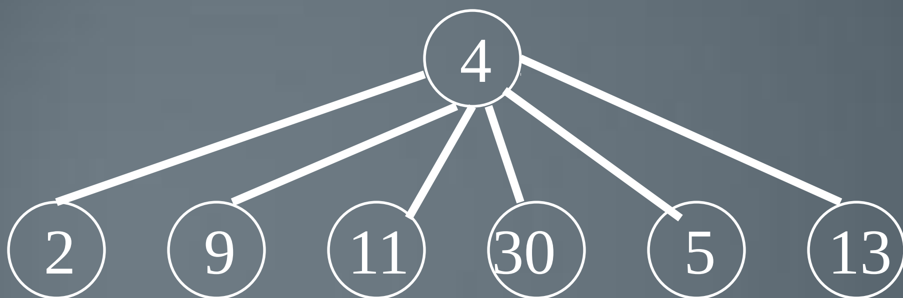
- A simple data structure for implementing disjoint sets is the *up-tree*.

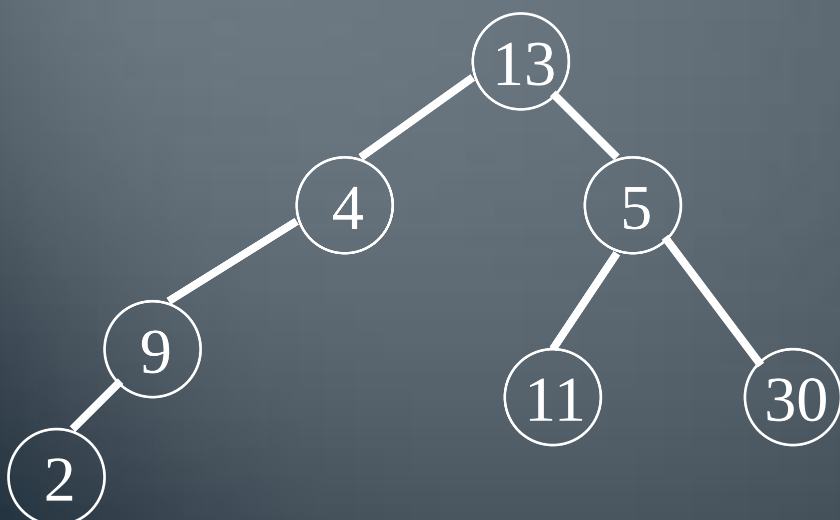
H, A and W belong to the same set. H is the representative

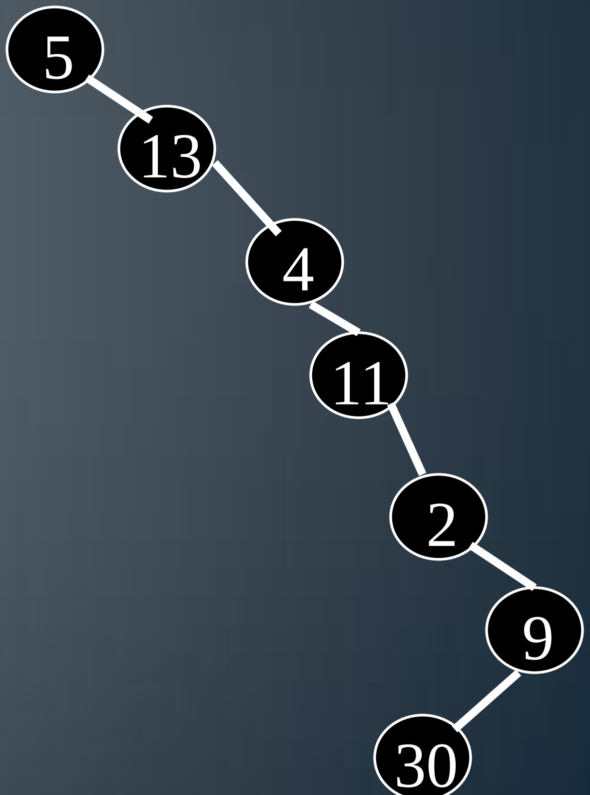
X, B, R and F are in the same set. X is the representative

A Set As A Tree

- S = {2, 4, 5, 9, 11, 13, 30}
- Some possible tree representations:







Operations in Up-Trees

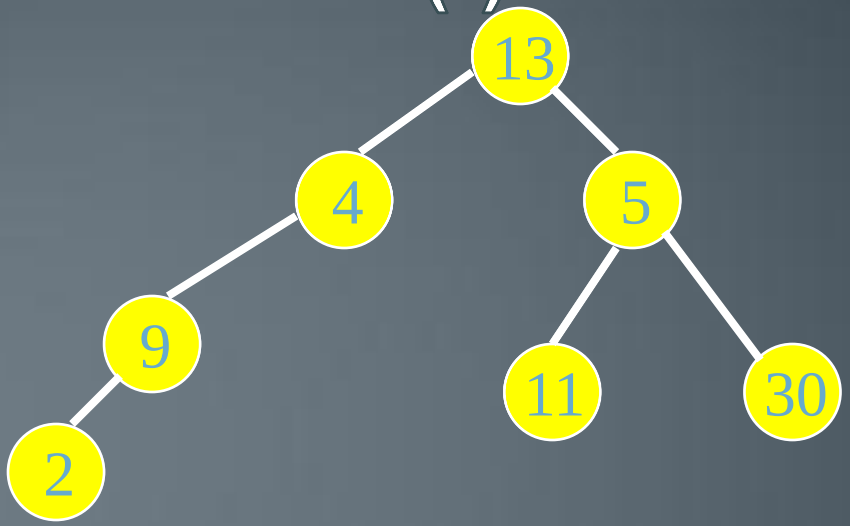
Find is easy. Just follow pointer to representative element. The representative has no parent.

find(x)

1. if (parent(x) exists)// not the root
   return(find(parent(x));
2. else return (x);

Worst case, height of the tree

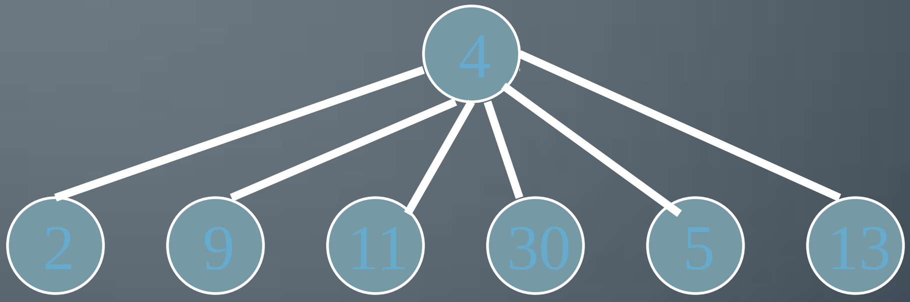
Steps For find(i)



- Start at the node that represents element i and climb up the tree until the root is reached
- Return the element in the root
- To climb the tree, each node must have a parent pointer

Result Of A Find Operation

- find(i) is to identify the set that contains element i
- In most applications of the union-find problem, the user does not provide set identifiers
- The requirement is that find(i) and find(j) return the same value iff elements i and j are in the same set

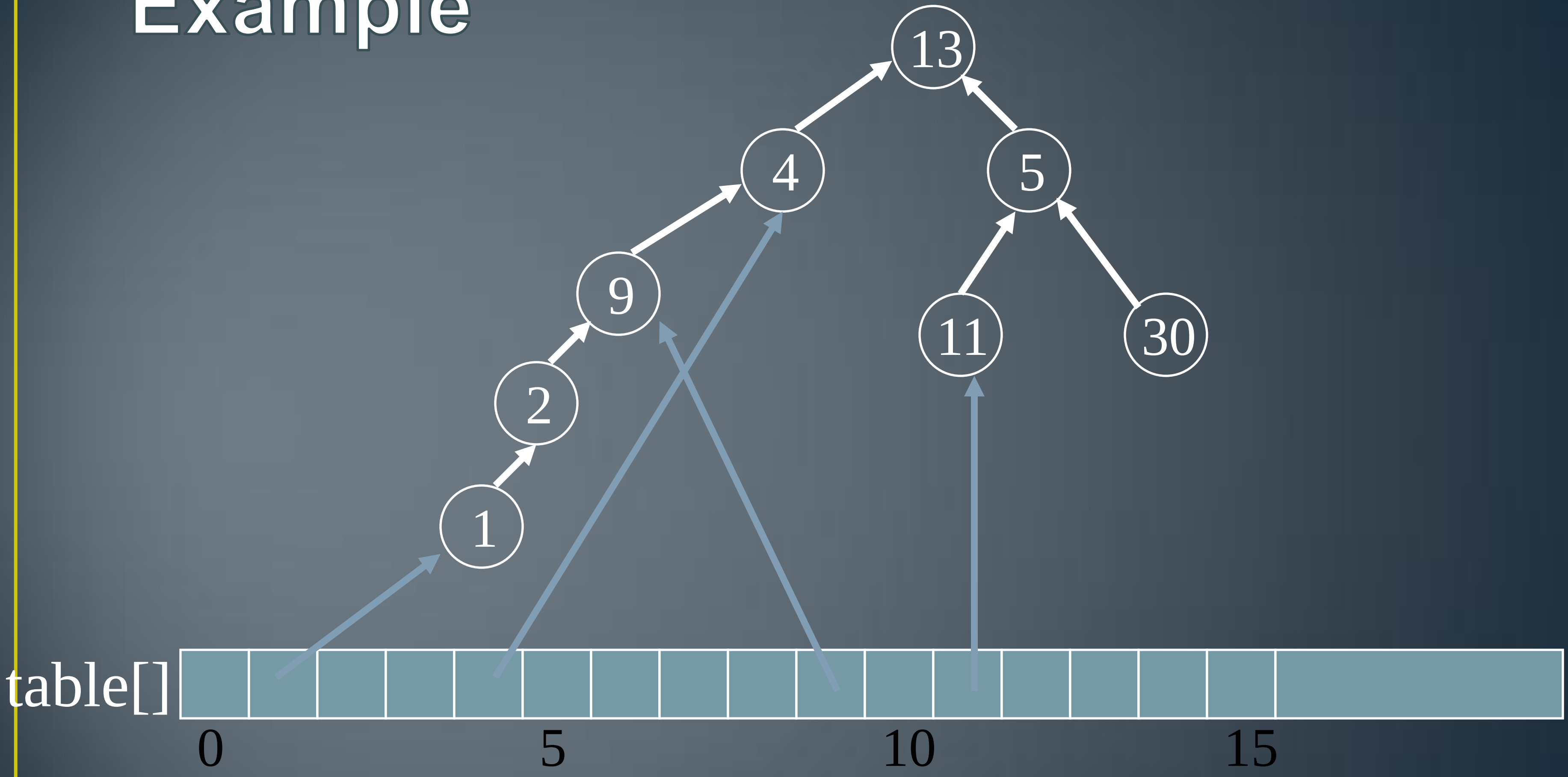


find(i) will return the element that is in the tree root

Possible Node Structure

- Use nodes that have two fields: element and parent
- Use an array table[] such that table[i] is a pointer to the node whose element is i
- To do a find(i) operation, start at the node given by table[i] and follow parent fields until a node whose parent field is null is reached
- Return element in this root node

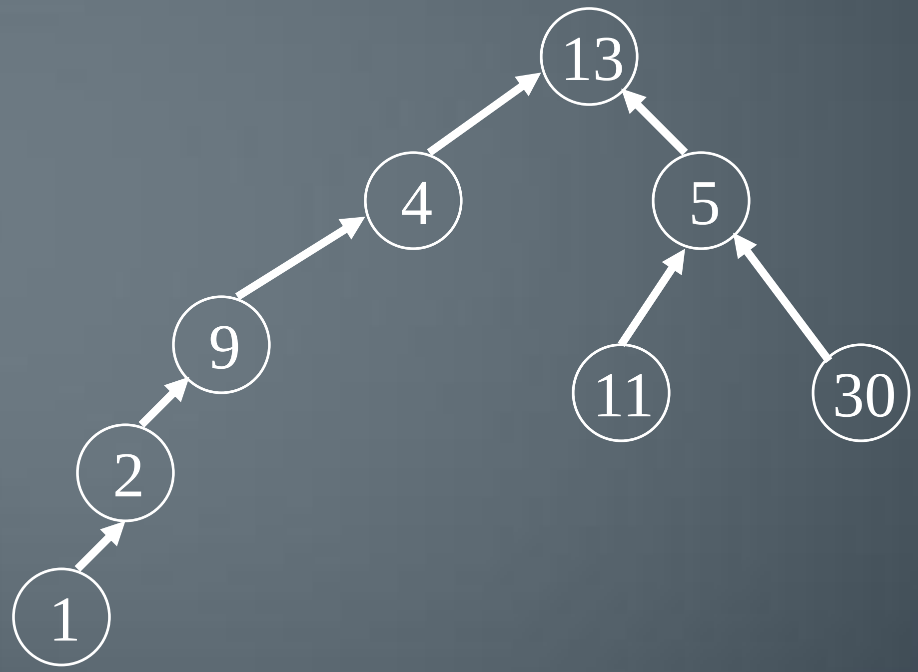
Example



(Only some table entries are shown.)

Better Representation

- Use an integer array parent[] such that parent[i] is the element that is the parent of element i

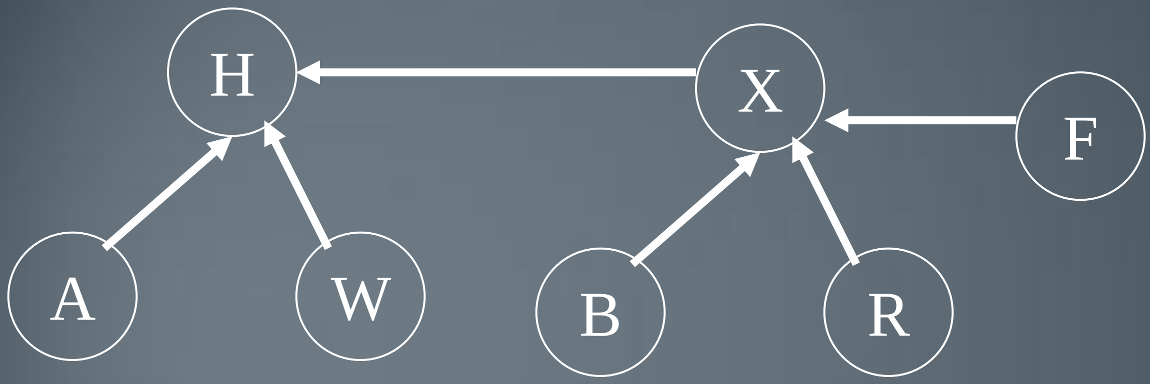


parent[]		2	9		13	13				4		5		0			
	0					5					10					15	

Union

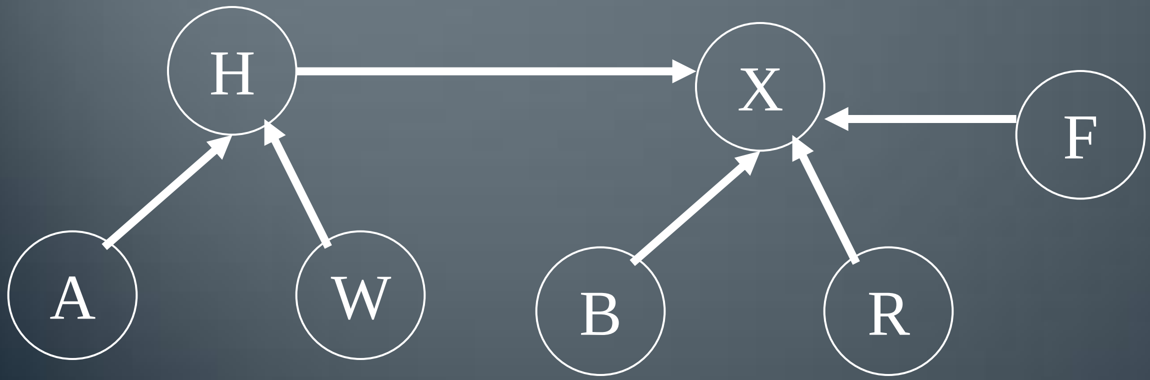
- Union is more complicated.
- Make one representative element point to the other, but which way?
  Does it matter?
- In the example, some elements are now deeper away from the root

Union(H, X)



X points to H

B, R and F are now deeper



H points to X

A and W are now deeper

Union

```
public union(rootA, rootB)
    {parent[rootB] = rootA;}
```

- Time Complexity: O(1)

A worse case for Union

Union can be done in O(1), but may cause find to become O(n)

Consider the result of the following sequence of operations:

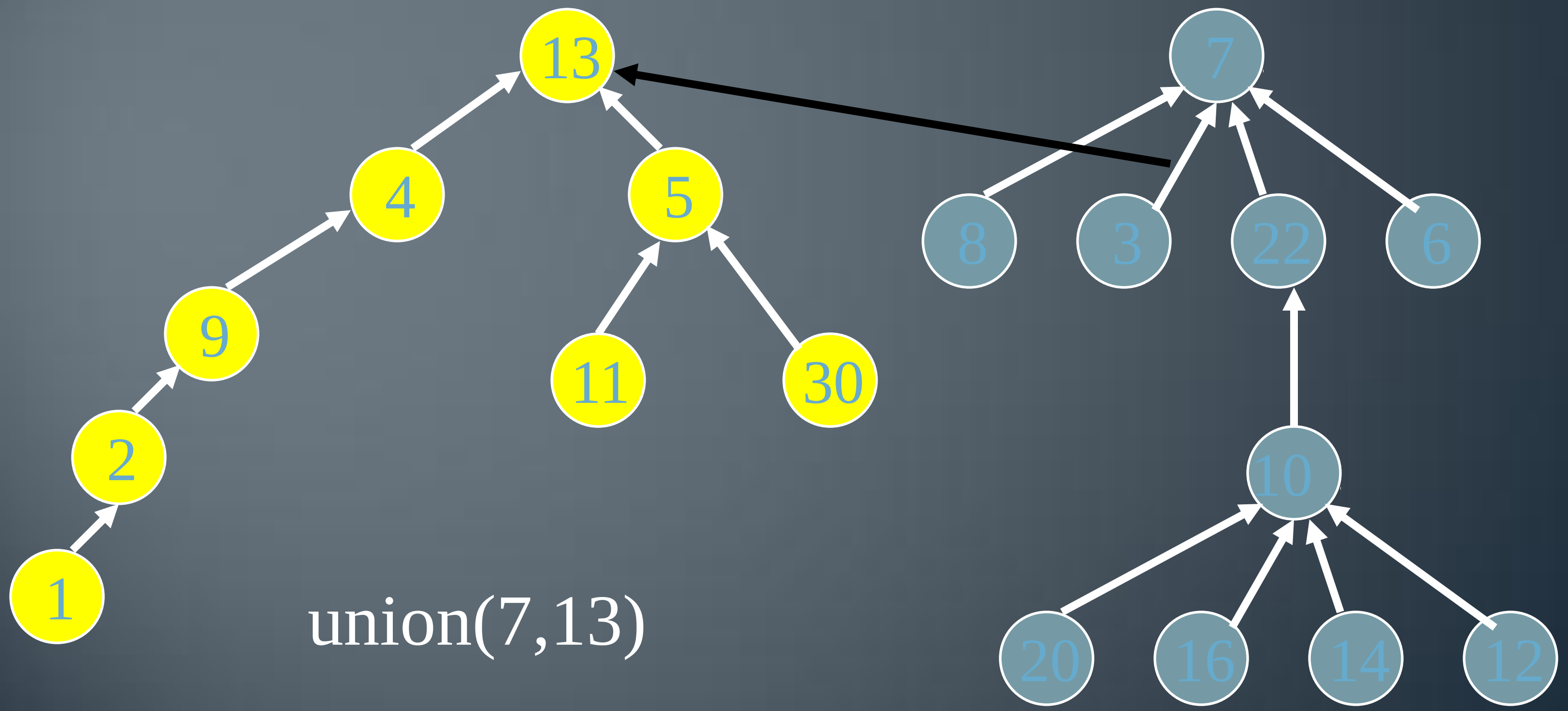
Union (A, B)
Union (C, A)
Union (D, C)
Union (E, D)

Two Heuristics

- There are two heuristics that improve the performance of union-find.
  - Union by weight or height
  - Path compression on find

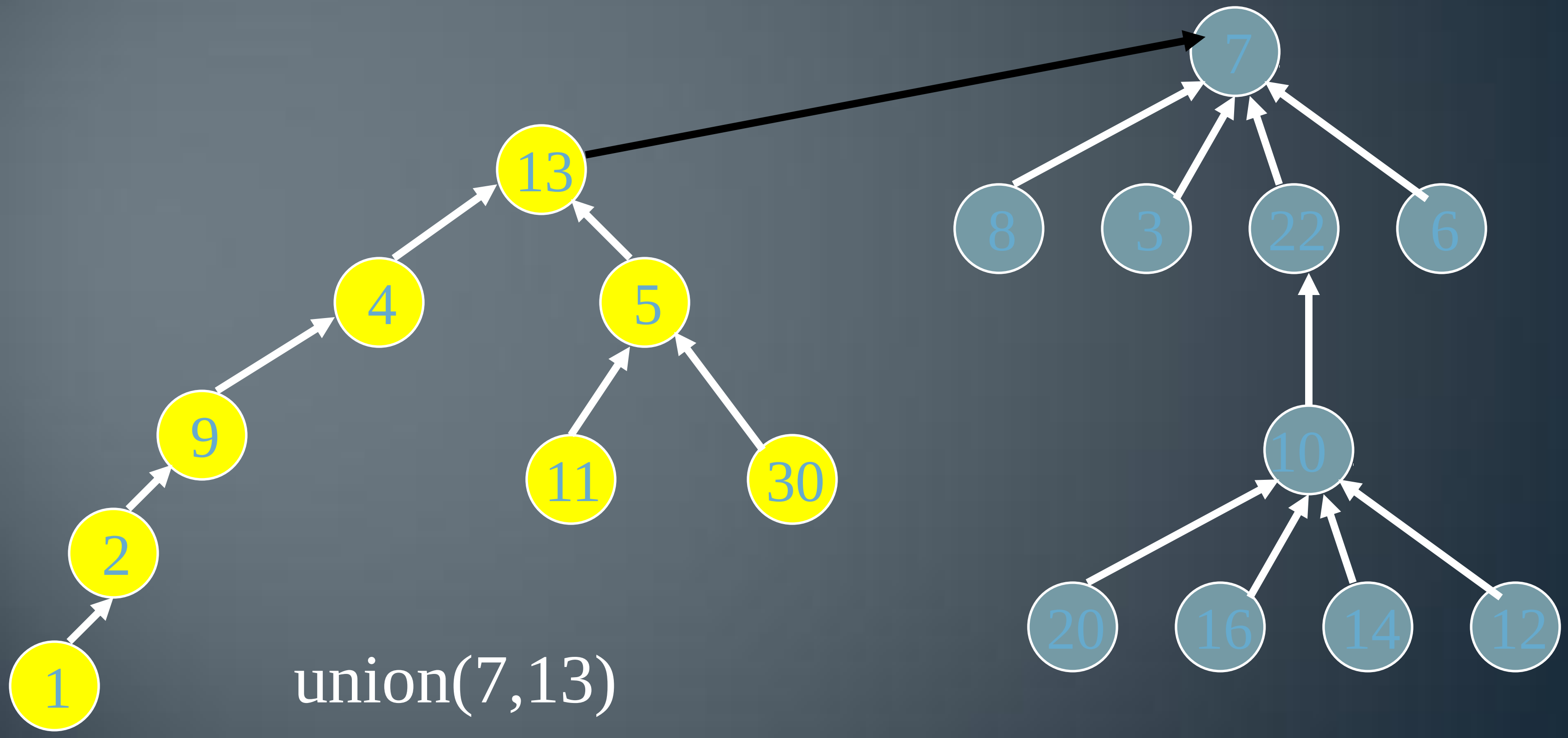
Height Rule

- Make tree with smaller height a subtree of the other tree
- Break ties arbitrarily



Weight Rule

- Make tree with fewer number of elements a subtree of the other tree
- Break ties arbitrarily



union(7,13)

Implementation

- Root of each tree must record either its height or the number of elements in the tree.
- When a union is done using the height rule, the height increases only when two trees of equal height are united.
- When the weight rule is used, the weight of the new tree is the sum of the weights of the trees that are united.

Height Of A Tree

- If we start with single element trees and perform unions using either the height or the weight rule. The height of a tree with p elements is at most floor $(\log_2 p) + 1$.
- Proof is by induction on p.

Union by Weight Heuristic

Always attach smaller tree to larger.

```
union(x,y)
    rep_x = find(x);
    rep_y = find(y);
    if (weight[rep_x] < weight[rep_y])
        A[rep_x] = rep_y;
        weight[rep_y] += weight[rep_x];
    else
        A[rep_y] = rep_x;
        weight[rep_x] += weight[rep_y];
```

Performance w/ Union by Weight

- If unions are done by weight, the depth of any element is never greater than log n + 1.
- Inductive Proof:
  - Initially, ever element is at depth zero.
  - When its depth increases as a result of a union operation (it’s in the smaller tree), it is placed in a tree that becomes at least twice as large as before (union of two equal size trees).
  - How often can each union be done? -- lg n times, because after at most lg n unions, the tree will contain all n elements.
- Therefore, find becomes O(log n) when union by weight is used -- even without path compression

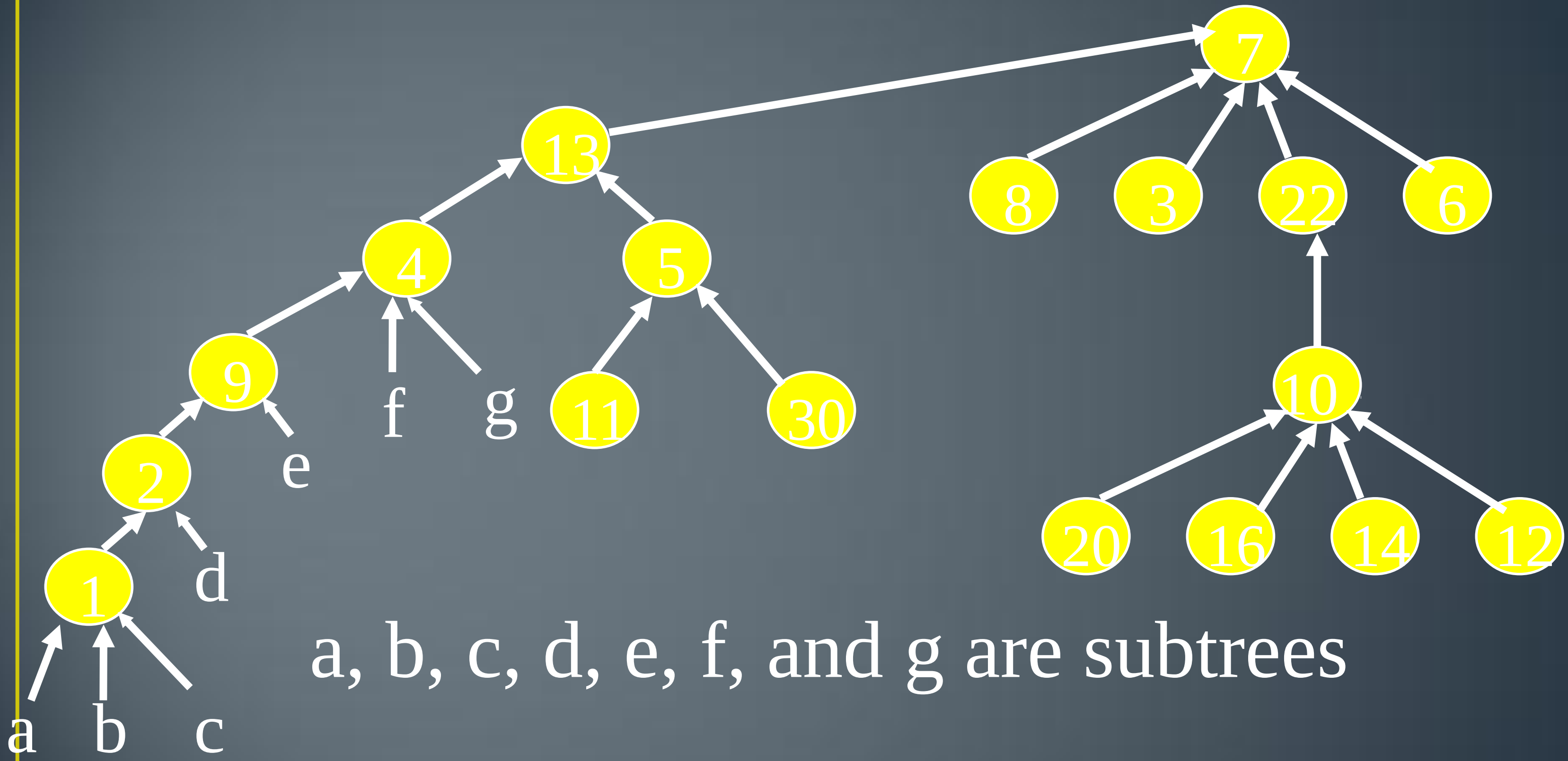
Path Compression

Each time we do a find on an element E, we make all elements on path from root to E be immediate children of root by making each element's parent be the representative.

```
find(x)
    if (A[x]<0)
          return(x);
    A[x] = find(A[x]);
    return (A[x]);
```

When path compression is done, a sequence of m operations takes O(m log n) time. Amortized time is O(log n) per operation.

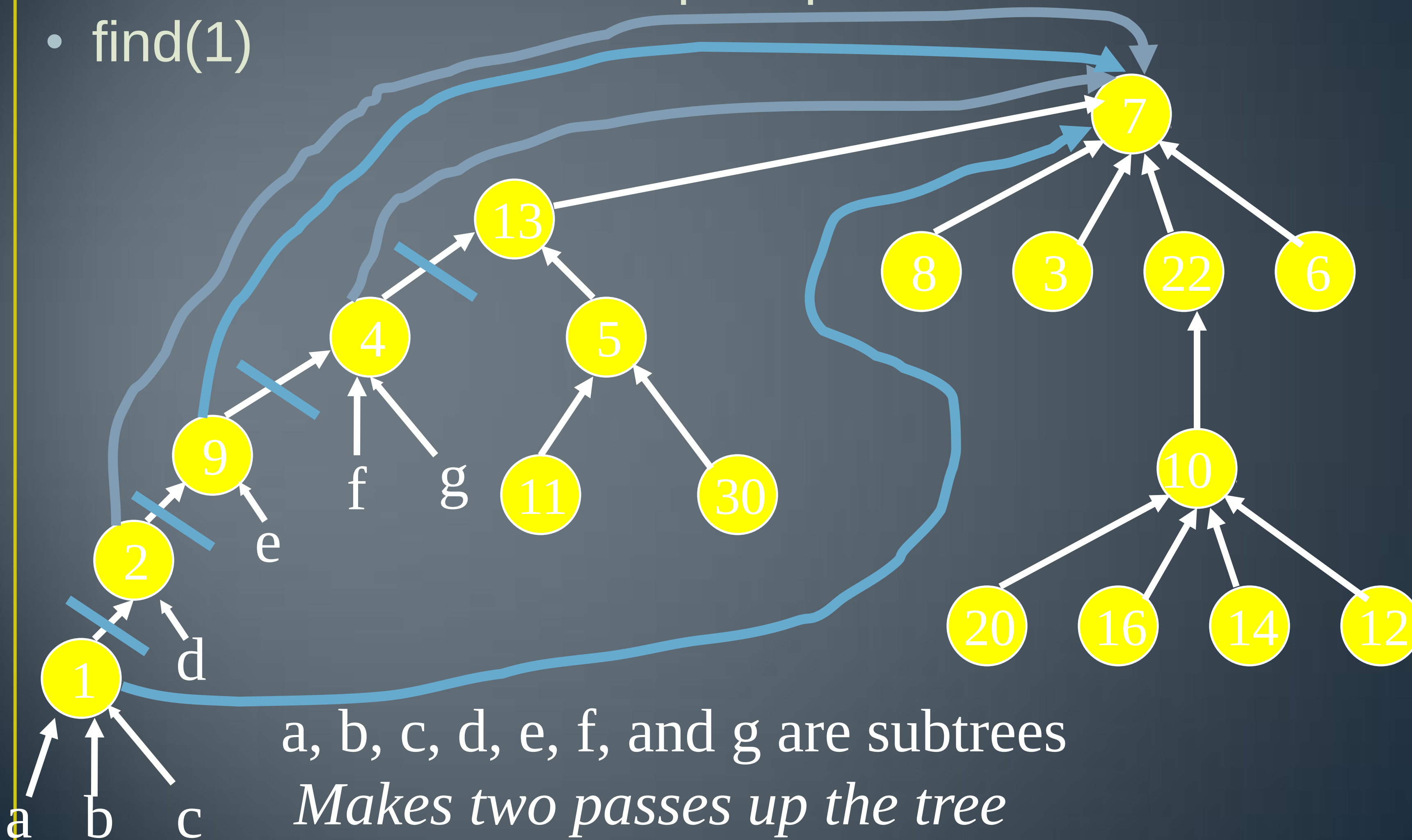
Path Compression



- find(1)
- Do additional work to make future finds easier

Path Compression

- Make all nodes on find path point to tree root.
- find(1)



a, b, c, d, e, f, and g are subtrees

Makes two passes up the tree

Ackermann’s Functions

- The Ackermann’s function is the simplest example of a well-defined total function which is computable but not primitive recursive.
- *"A function to end all functions" -- Gunter Dötzel.*
  - 1. If m = 0 then A(m, m) = m + 1
  - 2. If n = 0 then A(m, n) = A(m-1, 1)
  - 3. Otherwise, A(m, n) = A(m-1, A(m, n-1))
- The function f(n) = A(n, n) grows much faster than polynomials or exponentials or any function that you can imagine

Ackermann's Function

- Ackermann's function.
  - $A(m,n) = 2^n$, m = 1 and n >= 1
  - $A(m,n) = A(m-1,2)$, m>= 2 and n = 1
  - $A(m,n) = A(m-1,A(m,n-1))$, m,n >= 2
- Ackermann's function grows very rapidly as m and n increase
  - $A(2,4) = 2^{65,536}$

Time Complexity

- Inverse of Ackermann's function.
  - $\alpha(n) = \min\{k>=1 \mid A(k,1) > n\}$,
  - The inverse function grows very slowly
  - $\alpha(n) < 5$ until $n = 2^{A(4,1)} + 1$
  - $A(4,1) >> 10^{80}$
- For all practical purposes, $\alpha(n) < 5$

Time Complexity

Theorem 12.2 [Tarjan and Van Leeuwen]

Let T(n,m) be the maximum time required to process any intermixed sequence of n finds and unions.

$$T(n,m) = O(m \alpha (n))$$

when we start with singleton sets and use either the weight or height rule for unions and any one of the path compression methods for a find.