CAO: Lecture 1
Boolean Algebra
Introduction

Topics Covered

- Boolean Algebra
- Axioms
- Terminology
- N-bit boolean algebra
- Named theorems

Boolean Algebra

- We observed in our introduction that early in the development of computer hardware, a decision was made to use binary circuits because it greatly simplified the electronic circuit design.

- In order to work with binary circuits, it is helpful to have a conceptual framework to manipulate the circuits algebraically, building only the final "most simple" result.

- George Boole (1813-1864) developed a mathematical structure to deal with binary operations with just two values. Today, we call these structures *Boolean Algebras*.

Boolean Algebra Defined

- A Boolean Algebra B is defined as a 5-tuple {B, +, *, ’, 0, 1}
- + and * are *binary* operators,’ is a *unary* operator.
- The following axioms must hold for any elements a, b, c ∈ {0,1}

- *Axiom #1: Closure*
- If a and b are elements of B, (a + b) and (a * b) are in B.
- *Axiom #2: Cardinality*
- There are at least two elements a and b in B such that a != b.
- *Axiom #3: Commutative*
- If a and b are elements of B

  (a + b) = (b + a), and (a * b) = (b * a)

Axioms

Axiom #4: Associative

If a and b are elements of B

(a + b) + c = a + (b + c), and (a * b) * c = a * (b * c)

Axiom #5: Identity Element

B has identity elements with respect to + and *

0 is the identity element for +, and 1 is the identity element for *

a + 0 = a and a * 1 = a

Axiom #6: Distributive

* is distributive over + and + is distributive over *

a * (b + c) = (a * b) + (a * c), and a + (b * c) = (a + b) * (a + c)

Axiom #7: Complement Element

For every a in B there is an element a' in B such that

a + a' = 1, and a * a' = 0

Terminology

- Element 0 is called "FALSE".
- Element 1 is called "TRUE".
- '+' operation "OR", '*' operation "AND" and ' operation "NOT".
- Juxtaposition implies * operation: $ab = a * b$
- Operator order of precedence is: *()*, ', *, +.
  $a+bc = a+(b*c) \neq (a+b)*c$
  $ab' = a(b') \neq (a*b)'$
- Single Bit Boolean Algebra($1' = 0$ and $0' = 1$)

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	0	1

Proof by Truth Table

- Consider the distributive theorem: $a + (b * c) = (a + b)*(a + c)$. Is it true for a two bit Boolean Algebra?
- Can prove using a truth table. How many possible combinations of *a*, *b*, and *c* are there?
- Three variables, each with two values: $2*2*2 = 2^3 = 8$

a	*b*	*c*	*b*c*	*a+(b*c)*	*a+b*	*a+c*	*(a+b)*(a+c)*
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

n-bit Boolean Algebra

- Single bit Boolean Algebra can be extended to *n-bit* Boolean Algebra by define *sum(+)*, *product(*)* and *complement*(‘) as bit-wise operations
- Let a = 1101010, b = 1011011
- a + b = 1101010 + 1011011 = 1111011
- a * b = 1101010 * 1011011 = 1001010
- a’ = 1101010’ = 0010101

Principle of Duality

The dual of a statement S is obtained by interchanging * and +; 0 and 1.
Dual of (a*1)*(0+a’) = 0 is (a+0)+(1*a’) = 1
Dual of any theorem in a Boolean Algebra is also a theorem.
This is called the *Principle of Duality*.

Named Theorems

All of the following theorems can be proven based on the axioms. They are used so often that they have names.

Idempotent	$a + a = a$	$a * a = a$
Boundedness	$a + 1 = 1$	$a * 0 = 0$
Absorption	$a + (a*b) = a$	$a*(a+b) = a$
Associative	$(a+b)+c=a+(b+c)$	$(a*b)*c=a*(b*c)$

The theorems can be proven for a two-bit Boolean Algebra using a truth table, but you must use the axioms to prove it in general for all Boolean Algebras.

More Named Theorems

Involution	$(a')' = a$	
DeMorgan's	$(a+b)' = a' * b'$	$(a*b)'=a' + b'$

DeMorgan's Laws are particularly important in circuit design. It says that you can get rid of a complemented output by complementing all the inputs and changing ANDs to ORs. (More about circuits coming up…)

Proof using Theorems

- Use the properties of Boolean Algebra to reduce *(x + y)(x + x)* to *x*. *Warning, make sure you use the laws precisely.*

(x + y)(x + x)	**Given**
(x + y)x	**Idempotent**
x(x + y)	**Commutative**
x	**Absorption**

Unlike truth tables, proofs using Theorems are valid for any boolean algebra, but just bits.

Sources

- Lipschutz, Discrete Mathematics
- Mowle, A Systematic Approach to Digital Logic Design