NETWORK THEORY

LECTURE 5

SECTION B

TOPIC COVERED :TWO PORT NETWORK

Properties of the S Matrix

❖For reciprocal networks, the S-matrix is symmetric.

$$\Rightarrow \quad [S]=[S]^T$$

Note :

If $[A][B]=[U]$

then

$[B][A]=[U]$

❖For lossless networks, the S-matrix is unitary.

$$\Rightarrow \quad [S]^T[S]^* = [S]^*[S]^T = [U]$$

Identity matrix

Equivalently,

$$[S]^{T*} = [S]^{-1}$$

Notation: $[S]^\dagger = [S]^H = [S]^{T*}$

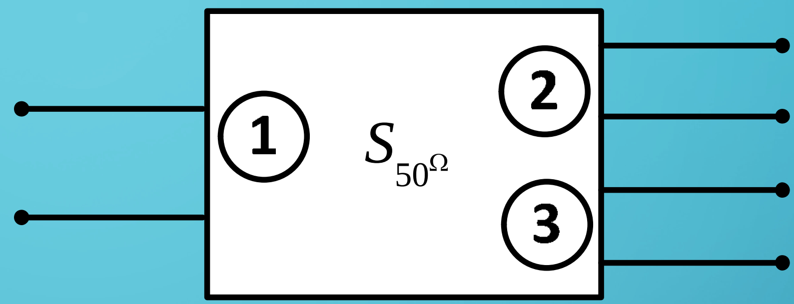
so $[S]^\dagger = [S]^{-1}$

N-port network

Take (i, j) element $\Rightarrow \sum_{k=1}^{N} S_{ik}^T S_{kj}^* = \sum_{k=1}^{N} S_{ki} S_{kj}^* = \delta_{ij}$ $\quad \delta_{ij} = \begin{cases} 1\,;\, i=j \\ 0\,;\, i\neq j \end{cases}$

Example

$$\left[S_{50\Omega}\right] = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$



Not unitary $\rightarrow$ Not lossless

(For example, column 2 doted with the conjugate of column three is not zero.

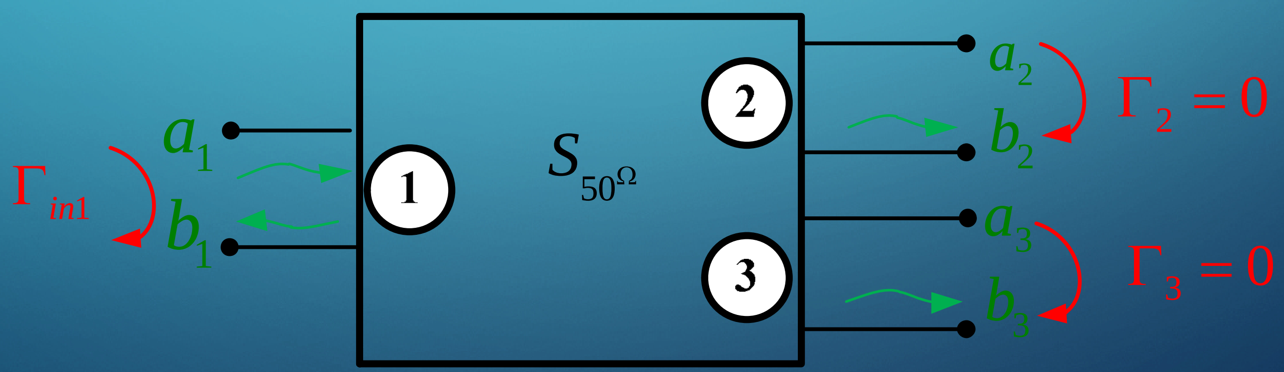
1) Find the input impedance looking into port 1 when ports 2 and 3 are terminated in 50 [Ω] loads.
2) Find the input impedance looking into port 1 when port 2 is terminated in a 75 [Ω] load and port 3 is terminated in a 50 [Ω] load.

Example (cont.)

1 If ports 2 and 3 are terminated in 50 [Ω]: $(a_2 = a_3 = 0)$

$$b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3$$

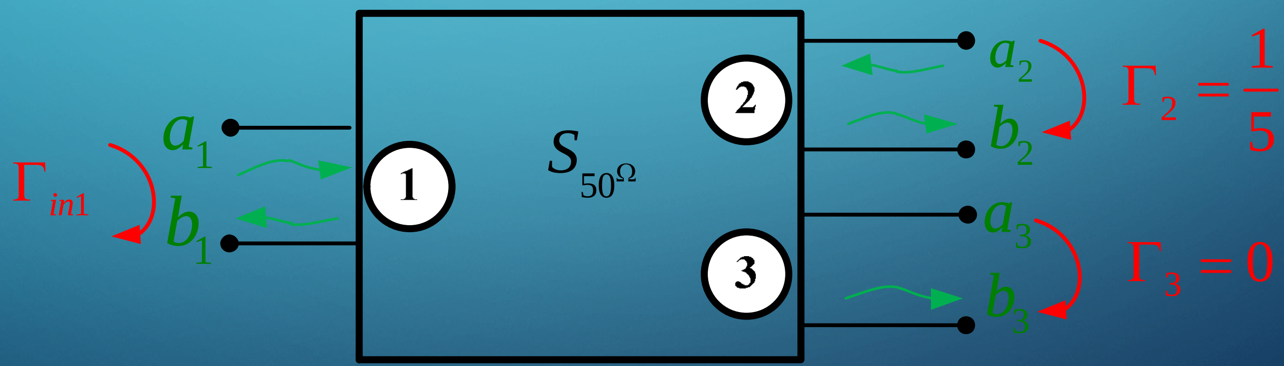
$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = S_{11} = 0 \quad \Rightarrow \quad Z_{in1} = 50[\Omega]$$



Example (cont.)

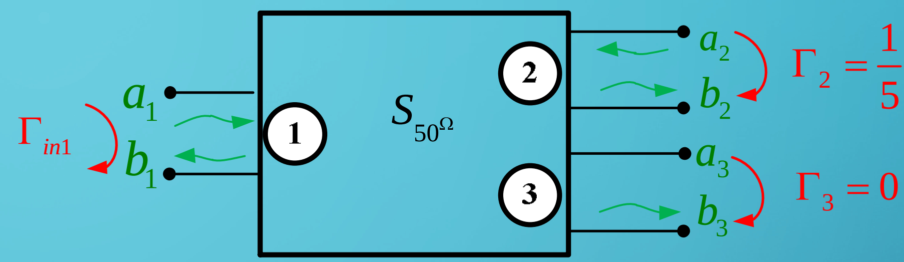
2) If port 2 is terminated in 75 [Ω] and port 3 in 50 [Ω]:

$$\Rightarrow \quad \Gamma_2 = \frac{a_2}{b_2} = \frac{75-50}{75+50} = \frac{1}{5}$$



Example (cont.)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = S_{11} + S_{12}\frac{a_2}{a_1} + S_{13}\frac{a_3}{a_1}$$

$$b_2 / a_1 = S_{21} + S_{22}\left(\frac{a_2}{a_1}\right) + S_{23}\left(\frac{a_3}{a_1}\right)$$

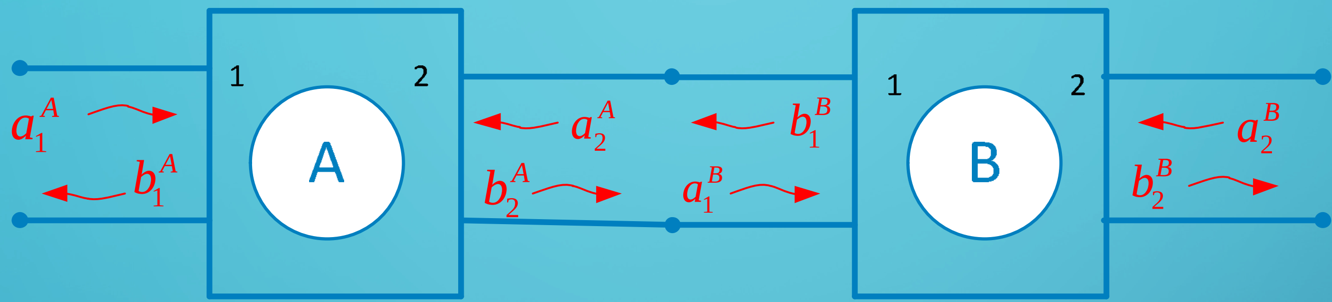
$$= S_{12}\left(\Gamma_2 \frac{b_2}{a_1}\right) = S_{12}\left(\Gamma_2 S_{21}\right) = \left(\frac{-j}{\sqrt{2}}\right)\left(\frac{1}{5}\right)\left(\frac{-j}{\sqrt{2}}\right) = -\frac{1}{10}$$

$$a_2 = \Gamma_2 b_2$$

$$\Rightarrow Z_{in1} = 50\left(\frac{1+\Gamma_{in_1}}{1-\Gamma_{in_1}}\right) = 44.55[\Omega]$$

Transfer (T) Matrix

For cascaded 2-port networks:



T Matrix:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$= [T] \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \dfrac{1}{S_{21}} & \dfrac{-S_{22}}{S_{21}} \\ \dfrac{S_{11}}{S_{21}} & S_{12} - \dfrac{S_{11}S_{22}}{S_{21}} \end{bmatrix}$$

$$[S] = \begin{bmatrix} \dfrac{-T_{21}}{T_{22}} & \dfrac{1}{T_{22}} \\ T_{11} - \dfrac{T_{12}^2}{T_{22}} & \dfrac{T_{12}}{T_{22}} \end{bmatrix}$$

(Derivation omitted)

Transfer (T) Matrix (cont.)

$$\Rightarrow \begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} b_2^A \\ a_2^A \end{bmatrix}$$

But $\begin{bmatrix} b_2^A \\ a_2^A \end{bmatrix} = \begin{bmatrix} a_1^B \\ b_1^B \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} a_1^B \\ b_1^B \end{bmatrix}$$

Hence $\begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \underbrace{\begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} T^B \end{bmatrix}}_{\begin{bmatrix} T^{AB} \end{bmatrix}} \begin{bmatrix} b_2^B \\ a_2^B \end{bmatrix}$

The T matrix of a cascaded set of networks is the product of the T matrices.

Conversion Between Parameters

TABLE 4.2 **Conversions Between Two-Port Network Parameters**

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\lvert Y\rvert}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\lvert Y\rvert}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\lvert Y\rvert}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\lvert Y\rvert}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\lvert Z\rvert}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\lvert Z\rvert}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\lvert Z\rvert}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\lvert Z\rvert}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\lvert Z\rvert}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\lvert Y\rvert}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

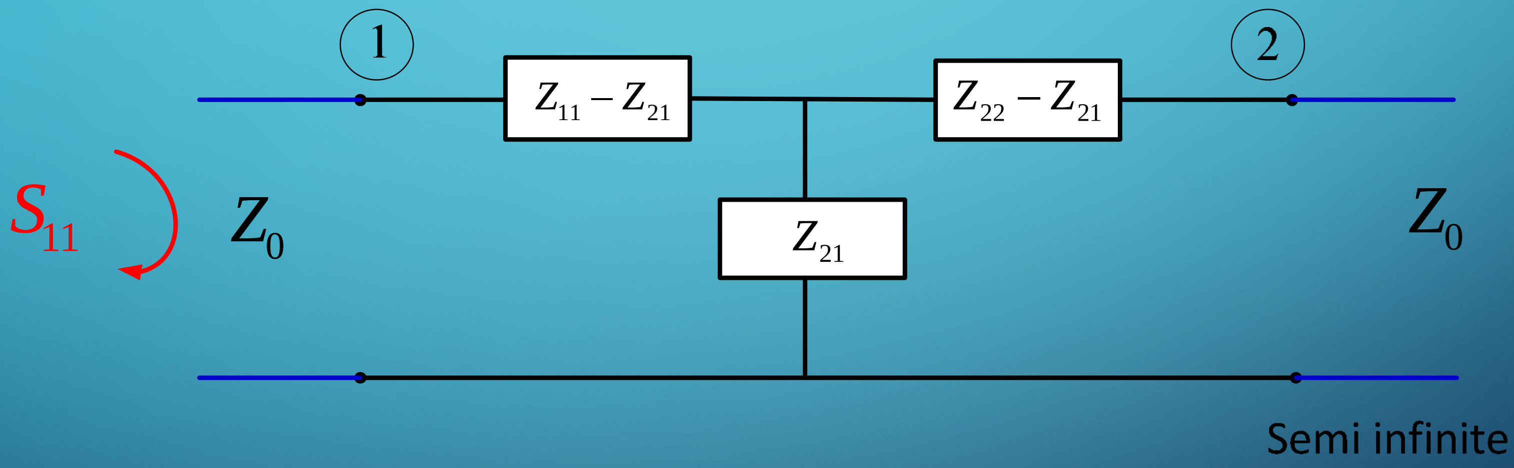
$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

Example

Derive S_{ij} from the Z parameters.

(The result is given inside row 1, column 2, of the previous table.)

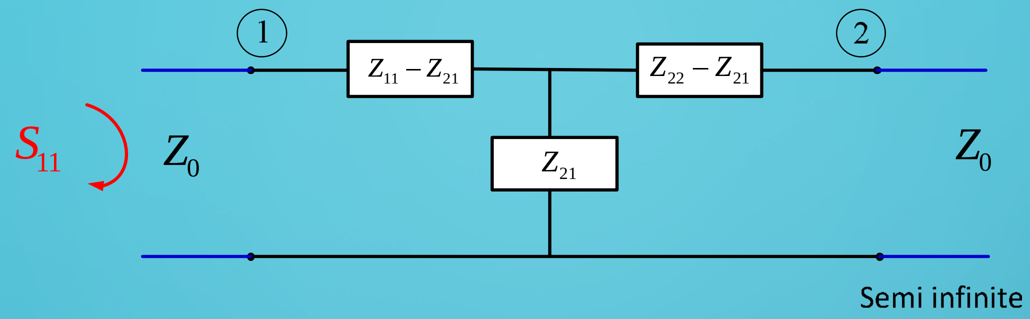
S_{11} Calculation:



$$S_{11} = \Gamma_{in1} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

$$Z_{in} = \left(Z_{11} - Z_{21}\right) + \left(Z_{21}\right) \,||\, \left[\left(Z_{22} - Z_{21}\right) + Z_0\right]$$

Example (cont.)



$$Z_{in} = (Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0 \right]$$

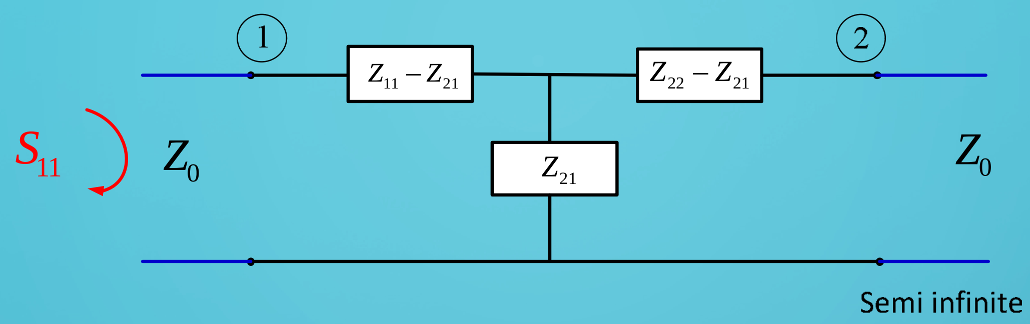
$$= (Z_{11} - Z_{21}) + \frac{Z_{21}(Z_{22} + Z_0 - Z_{21})}{Z_{22} + Z_0}$$

$$= \frac{(Z_{11} - Z_{21})(Z_{22} + Z_0) + Z_{21}(Z_{22} + Z_0 - Z_{21})}{Z_{22} + Z_0}$$

$$= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - Z_{21}Z_{22} - Z_{21}Z_0 + Z_{21}Z_{22} + Z_{21}Z_0 - Z_{21}^2}{Z_{22} + Z_0}$$

$$= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - Z_{21}^2}{Z_{22} + Z_0}$$

Example (cont.)



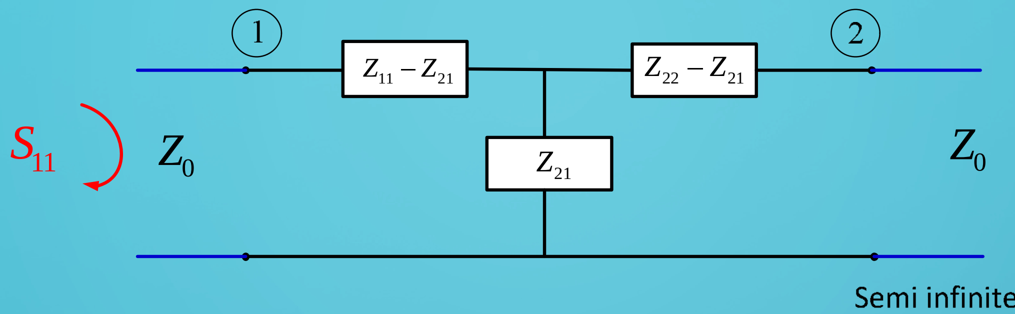
$$Z_{in} = \frac{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2}{Z_{22} + Z_0}$$

$$S_{11} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

so

$$S_{11} = \frac{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 - Z_0(Z_0 + Z_{22})}{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 + Z_0(Z_0 + Z_{22})}$$

Example (cont.)

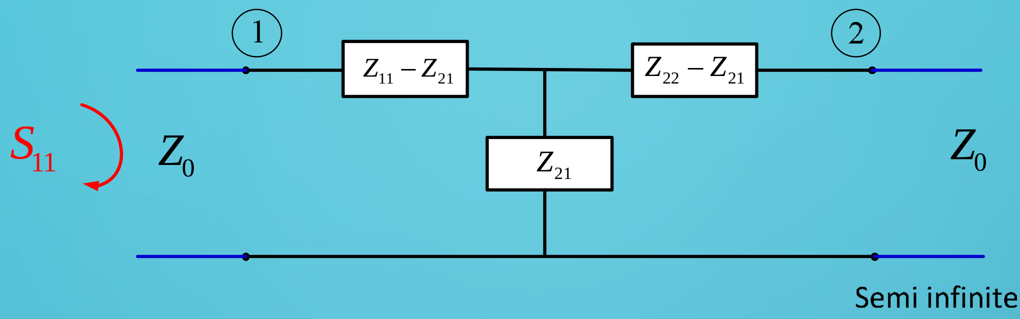


$$S_{11} = \frac{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 - Z_0(Z_0 + Z_{22})}{Z_{11}(Z_0 + Z_{22}) - Z_{21}^2 + Z_0(Z_0 + Z_{22})}$$

$$= \frac{Z_{11}Z_0 + Z_{11}Z_{22} - Z_{21}^2 - Z_0^2 - Z_0Z_{22}}{Z_{11}Z_0 + Z_{11}Z_{22} - Z_{21}^2 + Z_0^2 + Z_0Z_{22}}$$

$$= \frac{(Z_0 + Z_{22})(Z_{11} - Z_0) - Z_{21}^2}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

Example (cont.)



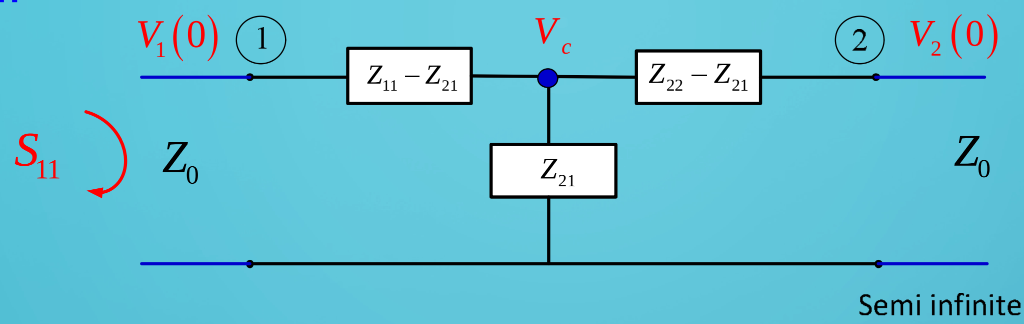
$$S_{11} = \frac{(Z_0 + Z_{22})(Z_{11} - Z_0) - Z_{21}^2}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

Note: to get S_{22}, simply let $Z_{11} \rightarrow Z_{22}$ in the previous result.

$$S_{22} = \frac{(Z_0 + Z_{11})(Z_{22} - Z_0) - Z_{21}^2}{(Z_0 + Z_{11})(Z_{22} + Z_0) - Z_{21}^2}$$

Example (cont.)

S_{21} Calculation:



Assume $V_1^+(0) = 1\ [\mathrm{V}]$

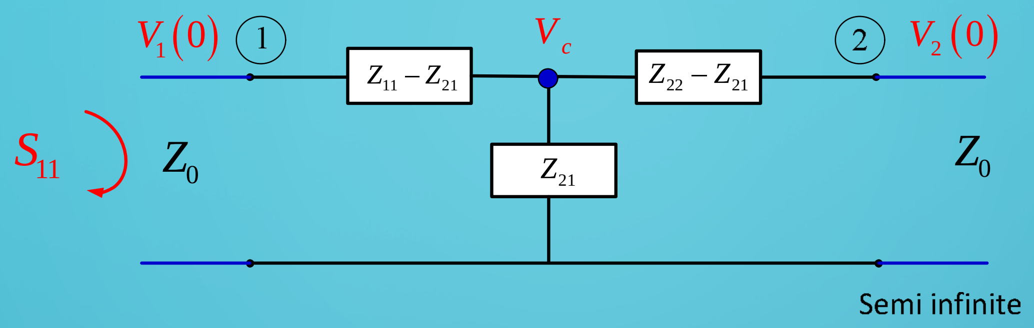
$$V_1(0) = 1 + S_{11} \qquad S_{21} = V_2^-(0) = V_2(0)$$

Use voltage divider equation twice:

$$V_c = V_1(0)\left(\frac{(Z_{21})\,||\left[(Z_{22}-Z_{21})+Z_0\right]}{(Z_{11}-Z_{21})+(Z_{21})\,||\left[(Z_{22}-Z_{21})+Z_0\right]}\right)$$

$$V_2(0) = V_c\left(\frac{Z_0}{(Z_{22}-Z_{21})+Z_0}\right)$$

Example (cont.)



Hence

$$S_{21} = \left(1 + S_{11}\right)\left(\frac{\left(Z_{21}\right) \,||\, \left[\left(Z_{22} - Z_{21}\right) + Z_0\right]}{\left(Z_{11} - Z_{21}\right) + \left(Z_{21}\right) \,||\, \left[\left(Z_{22} - Z_{21}\right) + Z_0\right]}\right)\left(\frac{Z_0}{\left(Z_{22} - Z_{21}\right) + Z_0}\right)$$

After simplifying, we should get the result in the table.

(You are welcome to check it!)