NETWORK THEORY

LECTURE 4

SECTION B

TOPIC COVERED :TWO PORT NETWORK

Scattering Parameters

$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0}$$

Output is matched ← input reflection coef. w/ output matched

$$S_{12} = \left.\frac{b_1(0)}{a_2(0)}\right|_{a_1=0}$$

Input is matched ← reverse transmission coef. w/ input matched

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

Output is matched ← forward transmission coef. w/ output matched

$$S_{22} = \left.\frac{b_2(0)}{a_2(0)}\right|_{a_1=0}$$

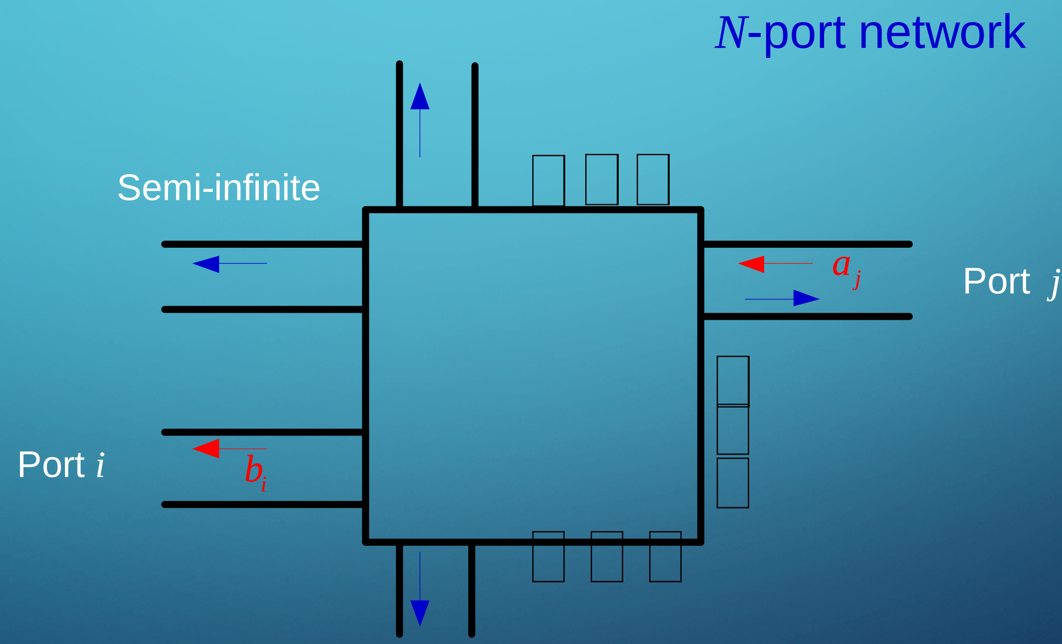
Input is matched ← output reflection coef. w/ input matched

Scattering Parameters (cont.)

For a general multiport network:

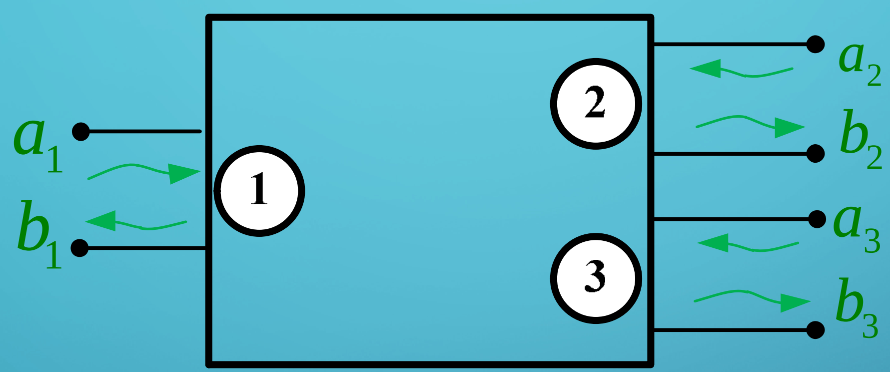
$$S_{ij} = \left. \frac{b_i(0)}{a_j(0)} \right|_{a_k = 0 \;\; k \neq j}$$

All ports except j are semi-infinite (or matched)



Scattering Parameters (cont.)

Illustration of a three-port network



Scattering Parameters (cont.)

For reciprocal networks, the *S*-matrix is symmetric.

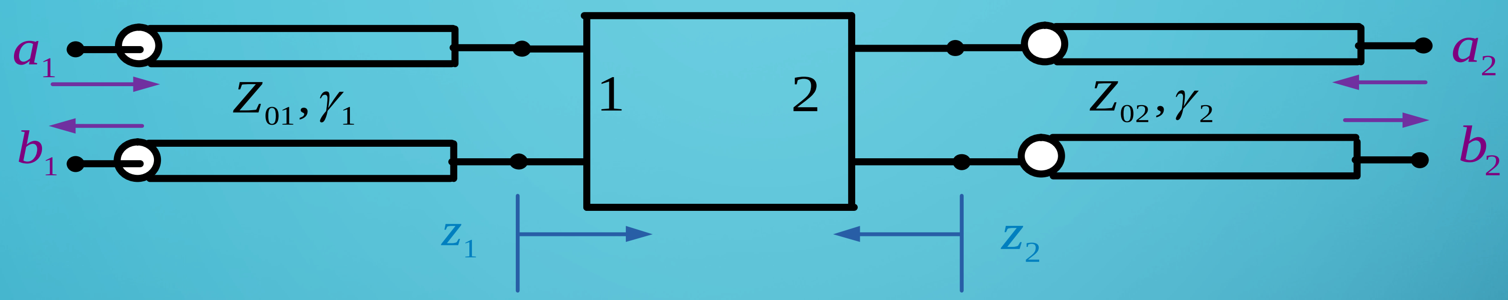
$$\Rightarrow S_{ij} = S_{ji} \qquad i \neq j$$

Note: If all lines entering the network have the same characteristic impedance, then

$$S_{ij} = \left. \frac{V_i^-(0)}{V_j^+(0)} \right|_{V_k^+=0 \;\; k \neq j}$$

Scattering Parameters (cont.)

Why are the wave functions (a and b) defined as they are?



$$P_i^+(0) = \frac{1}{2}\operatorname{Re}\left[V_i^+(0)\, I_i^{+*}(0)\right] = \frac{1}{2}\frac{\left|V_i^+(0)\right|^2}{Z_{0i}}$$

(assuming lossless lines)

Note:

$$a_i(0) = V_i^+(0)\Big/\sqrt{Z_{0i}}$$

$$\Rightarrow P_i^+(0) = \frac{1}{2}\left|a_i(0)\right|^2$$

Scattering Parameters (cont.)

Similarly,

$$P_i^-(0) = \frac{1}{2}\frac{\left|V_i^-(0)\right|^2}{Z_{0i}} = \frac{1}{2}\left|b_i(0)\right|^2$$

Also,

$$V_i^+(-l_i) = V_i^+(0)\, e^{+\gamma_i l_i}$$

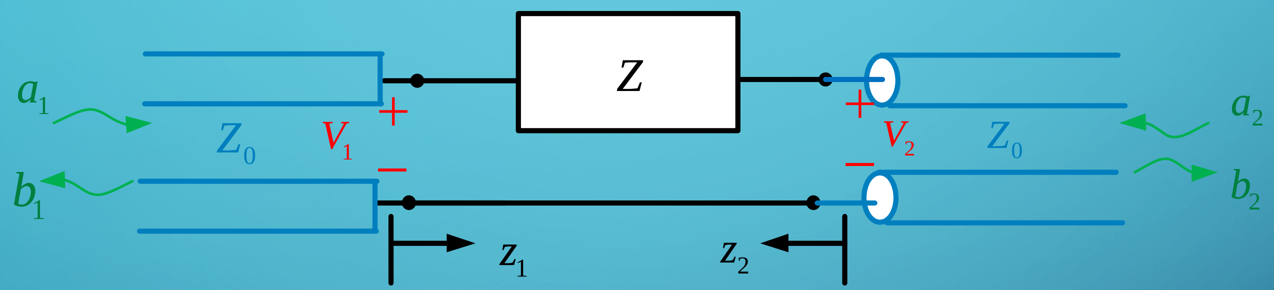
$$V_i^-(-l_i) = V_i^-(0)\, e^{-\gamma_i l_i}$$

$$\Rightarrow\ P_i^+(-l_i) = \frac{1}{2}\left|a_i(-l_i)\right|^2 = \frac{1}{2}\left|a_i(0)\right|^2 e^{+2\alpha_i l_i}$$

$$P_i^-(-l_i) = \frac{1}{2}\left|b_i(-l_i)\right|^2 = \frac{1}{2}\left|b_i(0)\right|^2 e^{-2\alpha_i l_i}$$

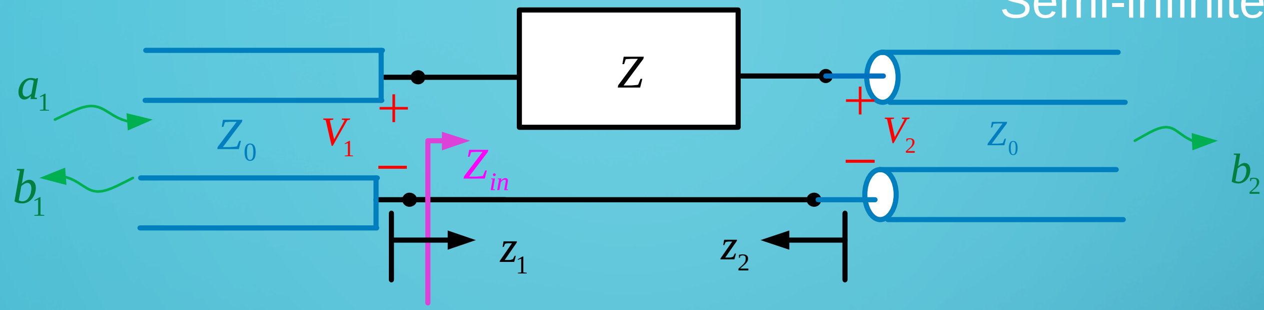
Example

Find the S parameters for a series impedance Z.



Note that two different coordinate systems are being used here!

Example (cont.)



S_{11} Calculation:

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0} = \frac{V_1^-(0)}{V_1^+(0)} = \left.\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right|_{a_2=0} = \frac{(Z + Z_0) - Z_0}{(Z + Z_0) + Z_0}$$

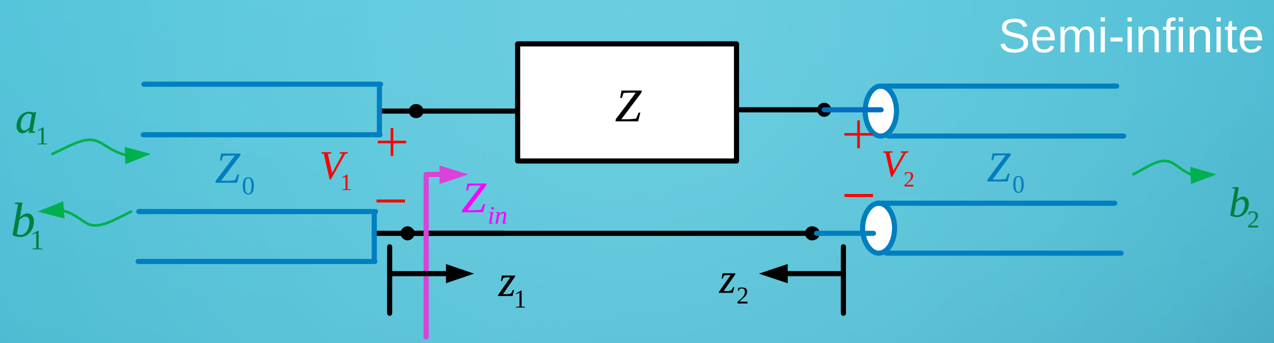
$$\Rightarrow \quad S_{11} = \frac{Z}{Z + 2Z_0}$$

By symmetry:

$$S_{22} = S_{11}$$

Example (cont.)

S_{21} Calculation:



$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

$$= \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0}$$

$$V_1^+(0) = a_1(0)\sqrt{Z_0}$$

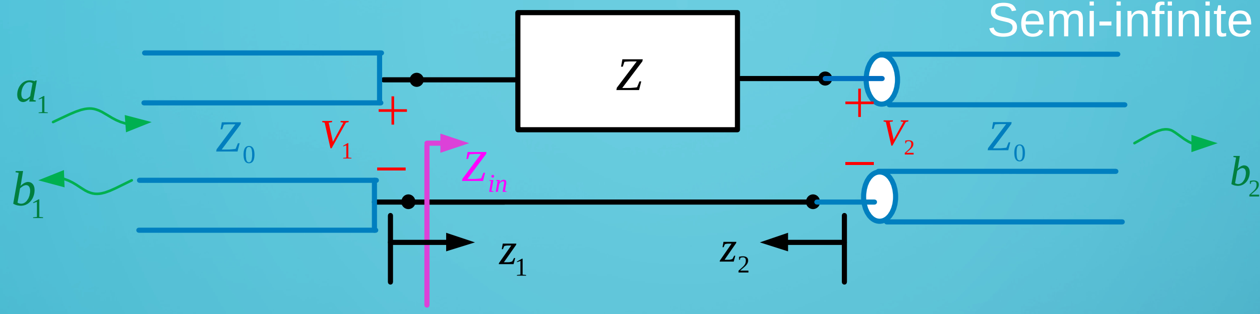
$$a_2 = 0 \Rightarrow V_2^-(0) = V_2(0)$$

$$V_2(0) = V_1(0)\left(\frac{Z_0}{Z+Z_0}\right)$$

$$V_1(0) = a_1\sqrt{Z_0}\,(1+S_{11})$$

$$\Rightarrow V_2^-(0) = V_2(0) = a_1\sqrt{Z_0}\,(1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right)$$

Example (cont.)



$$S_{21} = \frac{a_1(0)\sqrt{Z_0}\,(1+S_{11})\left(\dfrac{Z_0}{Z+Z_0}\right)}{a_1(0)\sqrt{Z_0}}$$

$$= (1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right) = \left(1+\frac{Z}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right) = \left(\frac{2Z+2Z_0}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right)$$

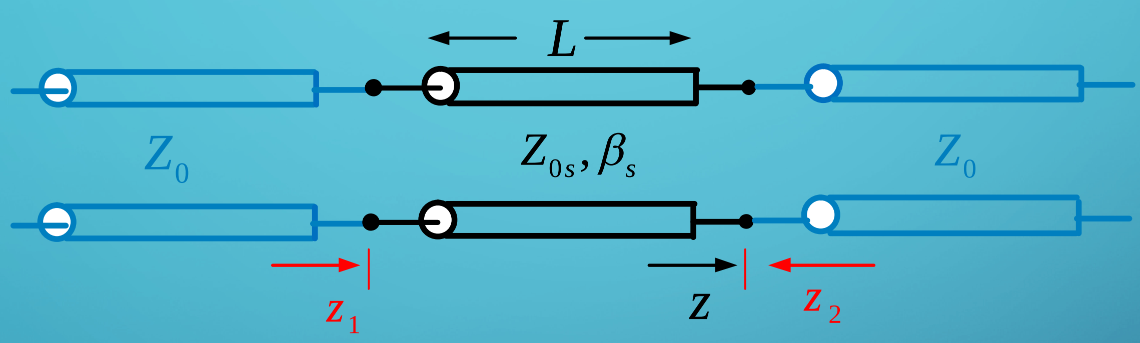
Hence

$$S_{21} = \frac{2\,Z_0}{Z+2\,Z_0}$$

$$S_{12} = S_{21}$$

Example

Find the S parameters for a length L of transmission line.

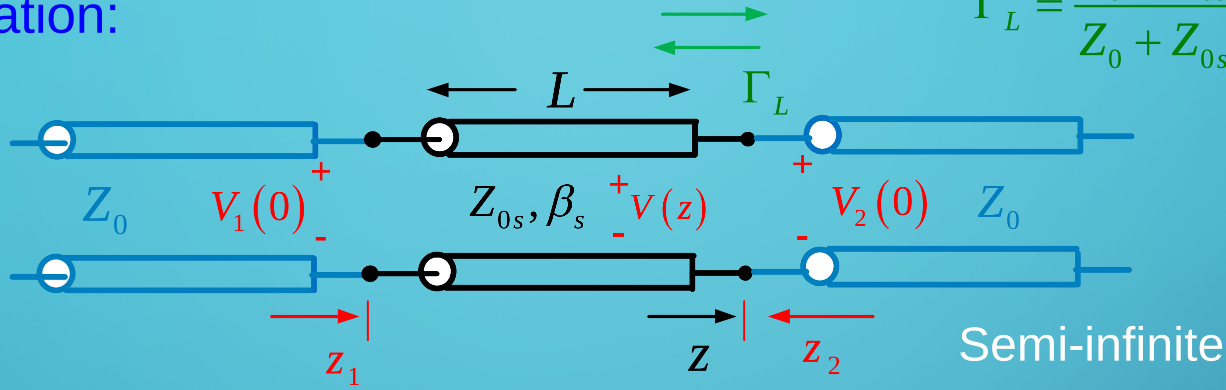


Note that three different coordinate systems are being used here!

Example (cont.)

S_{11} Calculation:

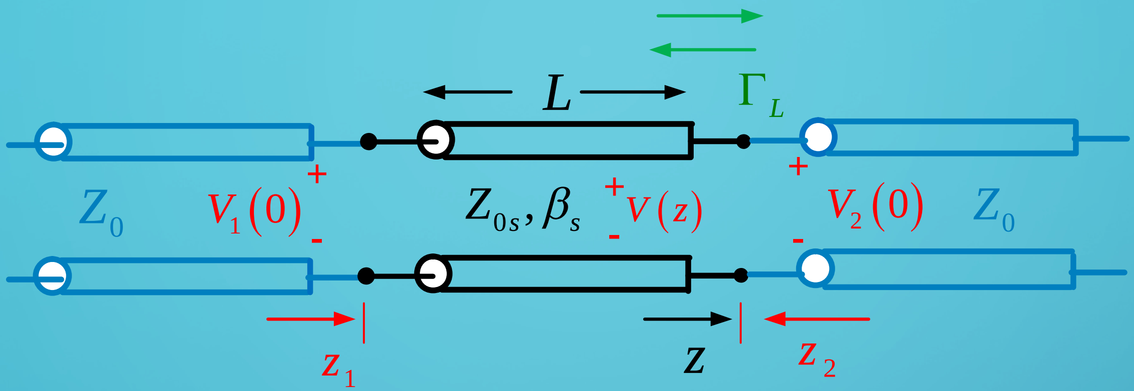
$$\Gamma_L = \frac{Z_0 - Z_{0s}}{Z_0 + Z_{0s}}$$



$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{Z_{in}\big|_{a_2=0} - Z_0}{Z_{in}\big|_{a_2=0} + Z_0} = S_{22} \text{ (by symmetry)}$$

$$Z_{in}\big|_{a_2=0} = Z_{0s}\frac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} = Z_{0s}\frac{\left(1+\Gamma_L e^{-j2\beta_s L}\right)}{\left(1-\Gamma_L e^{-j2\beta_s L}\right)}$$

Example (cont.)



Hence

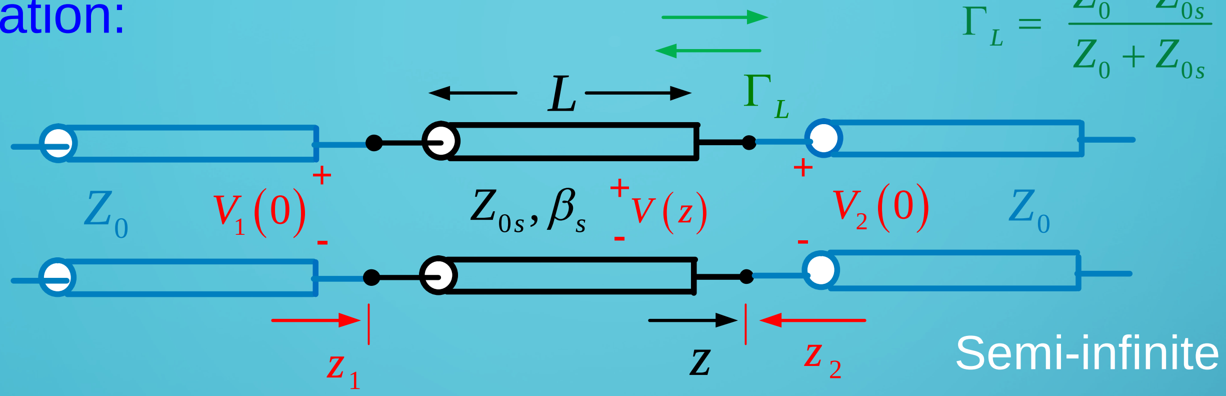
$$S_{11} = S_{22} = \frac{Z_{0s} \dfrac{\left(Z_0 + jZ_{0s} \tan \beta_s L\right)}{\left(Z_{0s} + jZ_0 \tan \beta_s L\right)} - Z_0}{Z_{0s} \dfrac{\left(Z_0 + jZ_{0s} \tan \beta_s L\right)}{\left(Z_{0s} + jZ_0 \tan \beta_s L\right)} + Z_0}$$

Note: If $Z_{0s} = Z_0 \Rightarrow Z_{in}\big|_{a_2=0} = Z_0 \Rightarrow S_{11} = S_{22} = 0$

Example (cont.)

S_{21} Calculation:

$$\Gamma_L = \frac{Z_0 - Z_{0s}}{Z_0 + Z_{0s}}$$



$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_2^-(0)/\sqrt{Z_0}}{V_1^+(0)/\sqrt{Z_0}}\right|_{a_2=0}$$

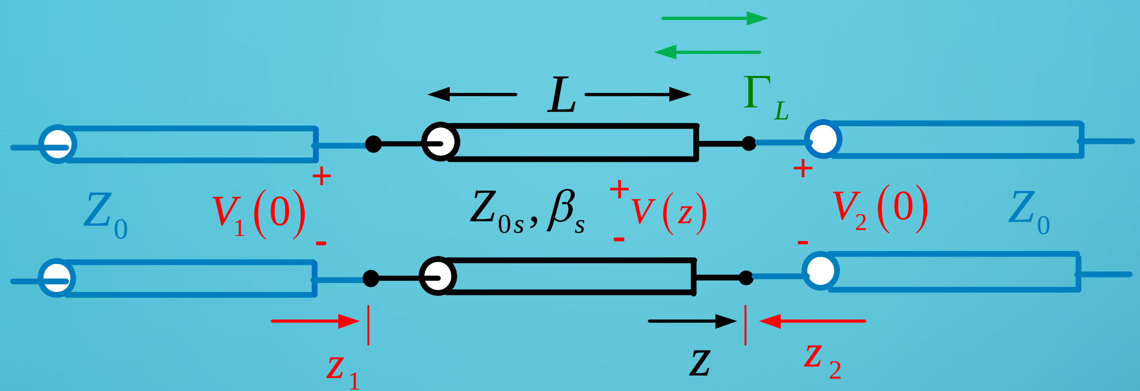
$$V_1(0) = V_1^+(0)(1+S_{11})$$

Hence, for the denominator of the S_{21} equation we have

$$V_1^+(0) = \frac{V_1(0)}{1+S_{11}}$$

We now try to put the numerator of the S_{21} equation in terms of V_1 (0).

Example (cont.)



$$V_2^-(0) = V_2(0) = V(0) = V^+(0)\left(1+\Gamma_L\right)$$

Next, use

$$V(z) = V^+(0)e^{-j\beta_s z}\left(1+\Gamma_L e^{+j2\beta_s z}\right)$$

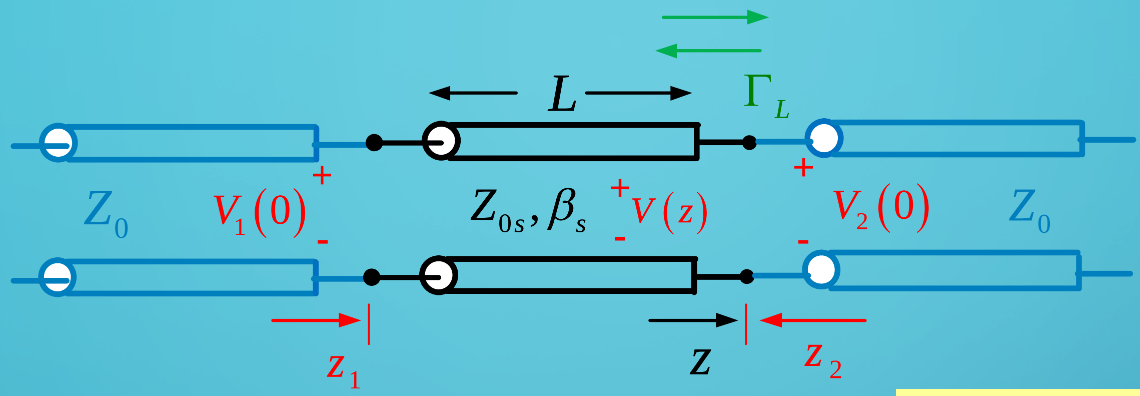
$$\Rightarrow V_1(0) = V(-L) = V^+(0)e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)$$

$$\Rightarrow V^+(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}$$

Hence, we have

$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}\left(1+\Gamma_L\right)$$

Example (cont.)



$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}\left(1+\Gamma_L\right)$$

$$V_1^+(0) = \frac{V_1(0)}{1+S_{11}}$$

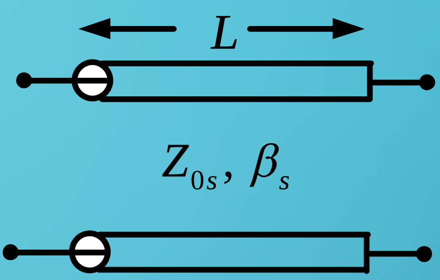
Therefore, we have

$$S_{21} = \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}}$$

so

$$S_{21} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}} = S_{12} \text{ by symmetry}$$

Example (cont.)



Special cases:

$$a)\quad Z_{0s} = Z_0 \Rightarrow S_{11} = S_{22} = 0, \quad \Gamma_L = 0$$

$$S_{21} = S_{12} = e^{-j\beta_s L}$$

$$[S] = \begin{bmatrix} 0 & e^{-j\beta_s L} \\ e^{-j\beta_s L} & 0 \end{bmatrix}$$

$$b)\quad L = \frac{\lambda_g}{2} \Rightarrow \beta_s L = \frac{2\pi}{\lambda_g}\frac{\lambda_g}{2} = \pi$$

$$\Rightarrow Z_{in}\big|_{a_2=0} = Z_0 \quad \Rightarrow S_{11} = S_{22} = 0$$

$$e^{-j\beta_s L} = -1 \quad \Rightarrow \quad S_{21} = -1$$

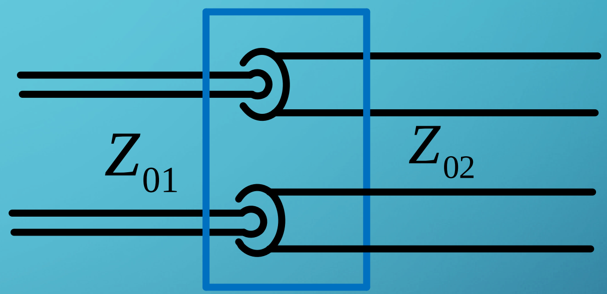
$$[S] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Example

Find the S parameters for a step-impedance discontinuity.

$$S_{11} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

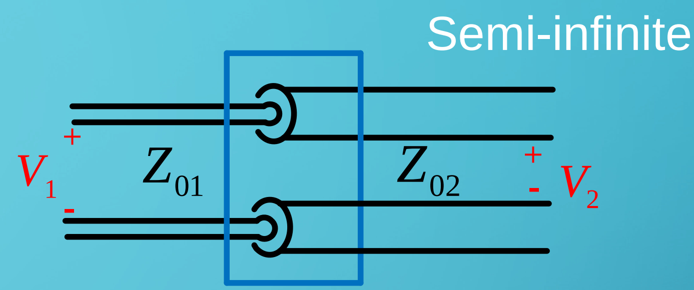
$$S_{22} = \frac{Z_{01} - Z_{02}}{Z_{02} + Z_{01}} = -S_{11}$$



$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0} = \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

Example (cont.)

S_{21} Calculation:



Because of continuity of the voltage across the junction, we have:

$$V_2^-(0)\Big|_{a_2=0} = V_2(0)\Big|_{a_2=0} = V_1(0)\Big|_{a_2=0} = V_1^+(0)(1+S_{11})$$

$$S_{21} = = \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0} = \left.\frac{\dfrac{V_1^+(0)(1+S_{11})}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

$$1+S_{11} = 1+\frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

$$= \frac{2Z_{02}}{Z_{02}+Z_{01}}$$

so

$$S_{21} = (1+S_{11})\sqrt{\frac{Z_{01}}{Z_{02}}}$$

Hence

$$S_{21} = S_{12} = 2\frac{\sqrt{Z_{01}Z_{02}}}{Z_{01}+Z_{02}}$$

Properties of the S Matrix (cont.)

Example:

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Unitary $\Rightarrow$

$$S_{11}S_{11}^{*} + S_{21}S_{21}^{*} + S_{31}S_{31}^{*} = 1$$
$$S_{12}S_{12}^{*} + S_{22}S_{22}^{*} + S_{32}S_{32}^{*} = 1$$
$$S_{13}S_{13}^{*} + S_{23}S_{23}^{*} + S_{33}S_{33}^{*} = 1$$
$$S_{11}S_{12}^{*} + S_{21}S_{22}^{*} + S_{31}S_{32}^{*} = 0$$
$$S_{11}S_{13}^{*} + S_{21}S_{23}^{*} + S_{31}S_{33}^{*} = 0$$
$$S_{12}S_{13}^{*} + S_{22}S_{23}^{*} + S_{32}S_{33}^{*} = 0$$

The column vectors form an orthogonal set.

The rows also form orthogonal sets (see the note on the previous slide).