DISCRETE STRUCTURE

Lecture-27

Introduction to Planar Graphs

Topics covered

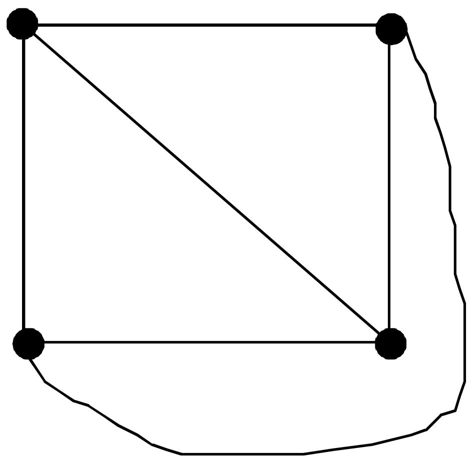
- ❑ Introduction to Planar graph
- ❑ Application

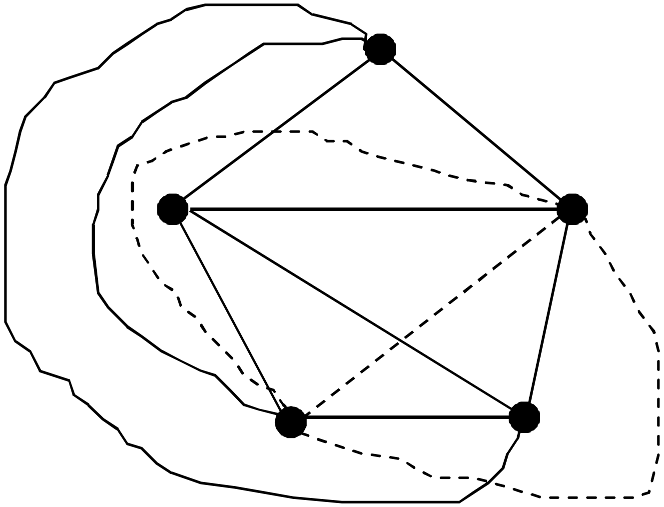
Introduction to Planar Graphs

Def1: A graph (or multigraph) *G* is called *planar* if *G* can be drawn in the plane with its edges intersecting only at vertices of *G*. Such a drawing of *G* is called an *embedding* of *G* in the plane.

Ex. K3 & k4 are planar, K_n for $n>4$ are nonplanar.

K3

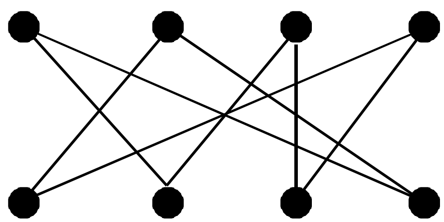


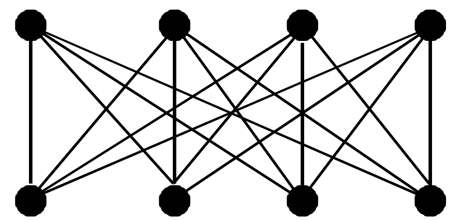


K4

Planar Graphs

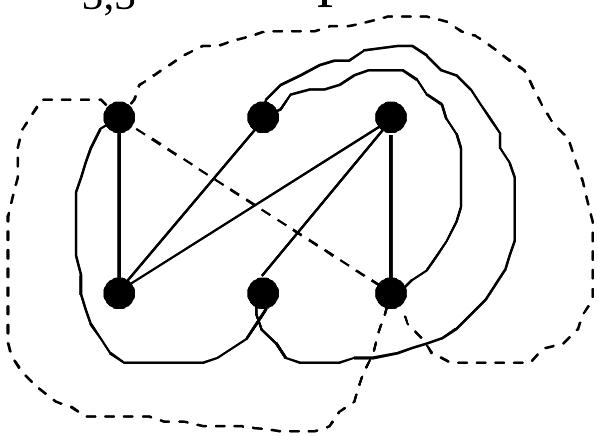
Def. 2 bipartite graph and complete bipartite graphs ($K_{m,n}$)





$K_{4,4}$

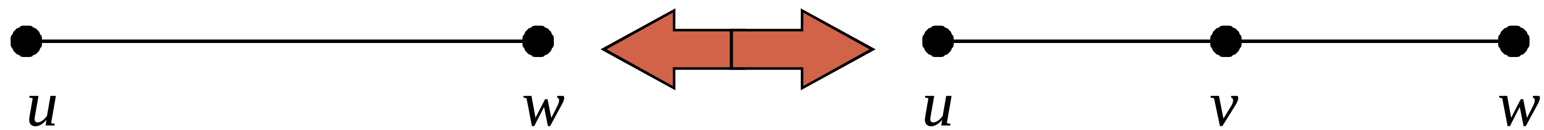
$K_{3,3}$ is not planar.



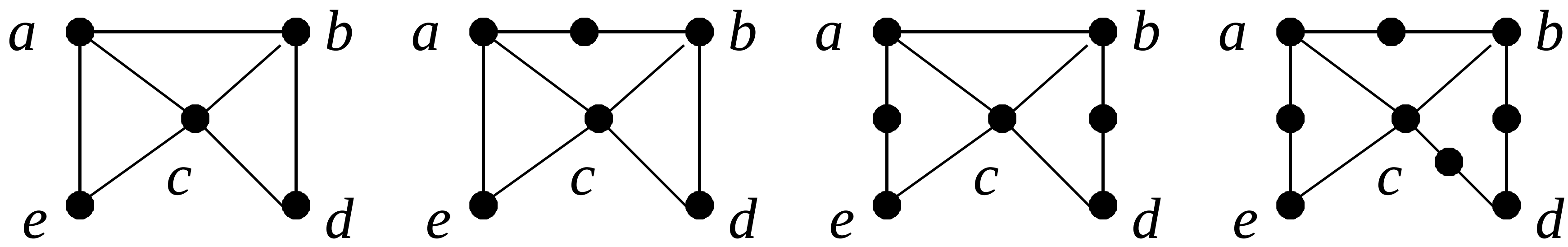
Therefore, any graph containing K_5 or $K_{4,4}$ is nonplanar.

11.4 Planar Graphs

Def. 3 *elementary subdivision* (*homeomorphic operation*)



G_1 and G_2 are called *homeomorphic* if they are isomorphic or if they can both be obtained from the same loop-free undirected graph H by a sequence of elementary subdivisions.

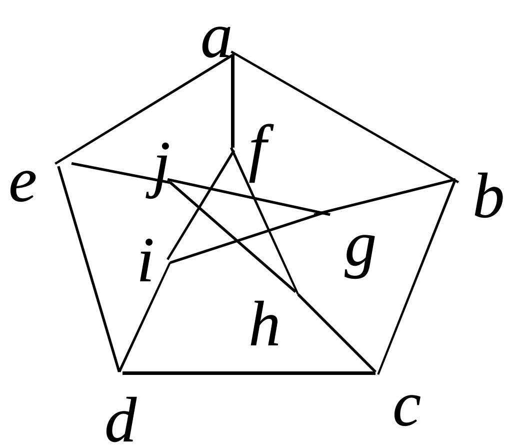


Two homeomorphic graphs are simultaneously planar or nonplanar.

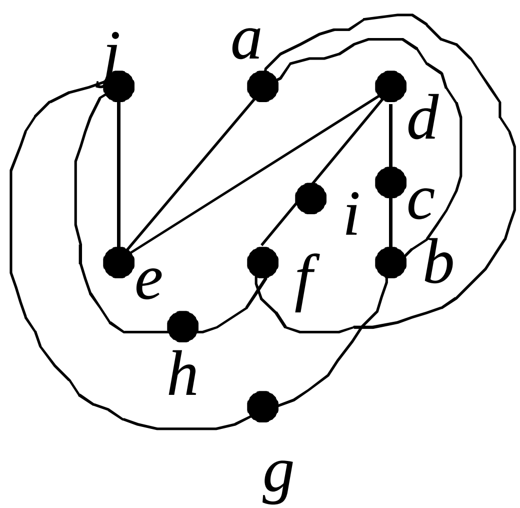
Planar Graphs

Theorem 1 (*Kuratowski's Theorem*) A graph is planar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

Ex. 1*Petersen graph*



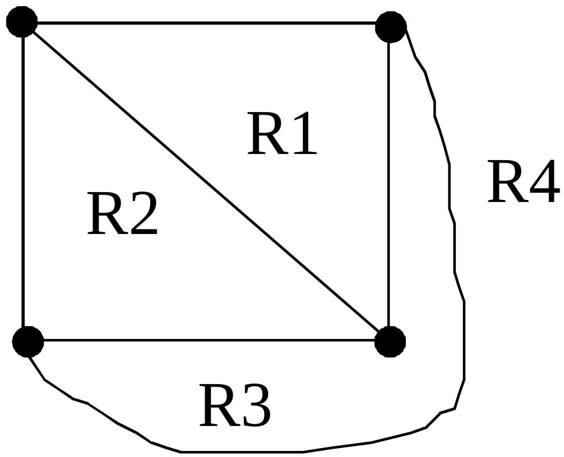
a subgraph homeomorphic to $K_{3,3}$



Petersen graph is nonplanar.

11.4 Planar Graphs

K_4



A planar graph divides the plane into several regions (faces), one of them is the infinite region.

v=4,e=6,r=4, v-e+r=2

Theorem 1 (*Euler's planar graph theorem*)

For a **connected** planar graph or multigraph:

$$v-e+r=2$$

number of vertices

number of edges

number of regions

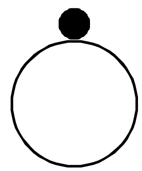
Planar Graphs

proof: The proof is by induction on *e*.

e=0 or 1

v-*e*+*r*=2

v=1
r=1
e=0



v=1
r=2
e=1

v=2
r=1
e=1

Assume that the result is true for any connected planar graph or multigraph with *e* edges, where $0 \le e \le k$

Now for *G*=(*V*,*E*) with |*E*|=*k*+1 edges, let *H*=*G*-(*a*,*b*) for *a*,b in *V*.

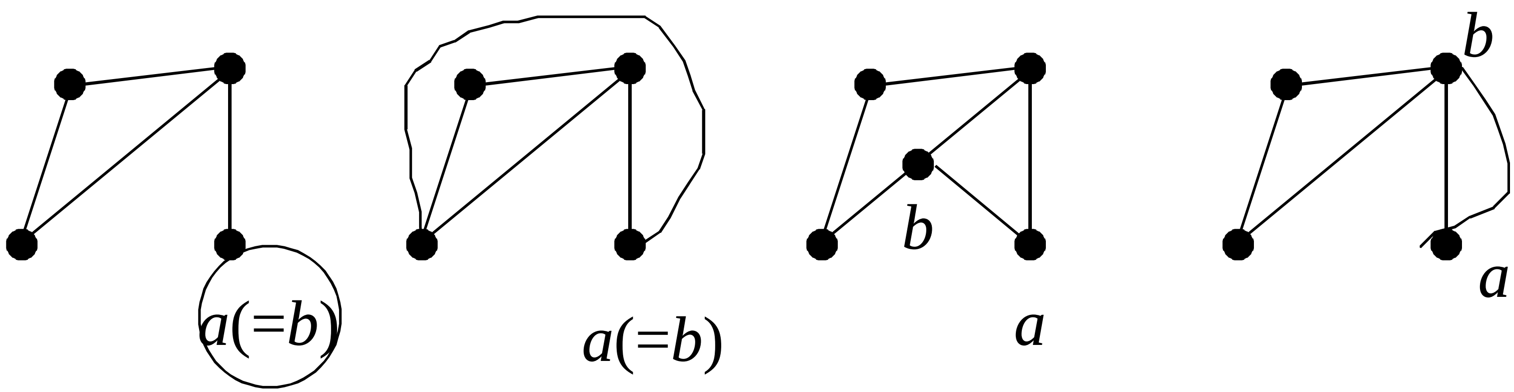
Since H has k edges, $v_H - e_H + r_H = 2$

And, $v_G = v_H, e_G = e_H - 1.$

Now consider the situation about regions.

Planar Graphs

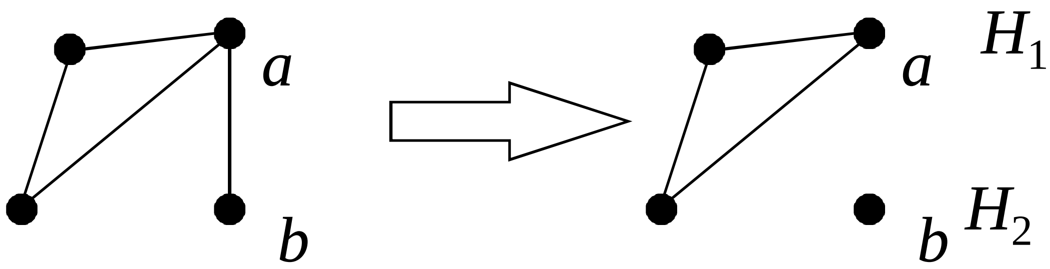
case 1: *H* is connected

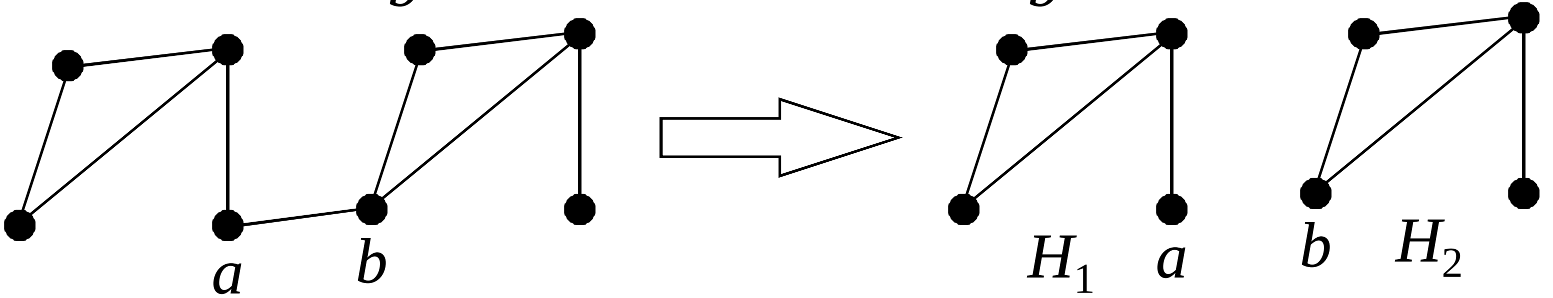


$$\therefore v_G - e_G + r_G = v_H - (e_H + 1) + (r_H + 1) = v_H - e_H + r_H = 2$$

Planar Graphs

case 2: *H* is disconnected





$v_{H_1} + v_{H_2} = v_G, e_{H_1} + e_{H_2} = e_G - 1, r_{H_1} + r_{H_2} = r_G + 1.$

And by the induction hypothesis, $v_{H_1} - e_{H_1} + r_{H_1} = 2,$

$v_{H_2} - e_{H_2} + r_{H_2} = 2.$ Therefore, $v_G - e_G + r_G = (v_{H_1} + v_{H_2})$

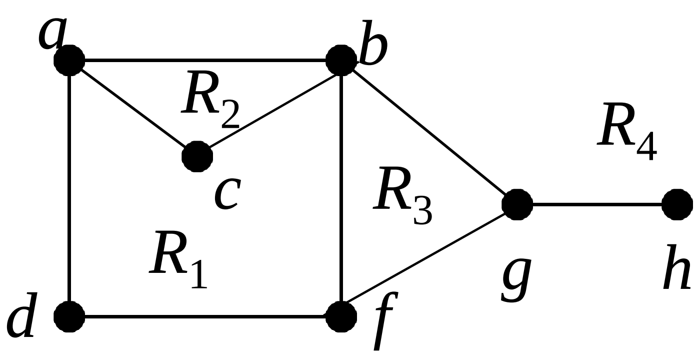
$-(e_{H_1} + e_{H_2} + 1) + (r_{H_1} + r_{H_2} - 1) = (v_{H_1} - e_{H_1} + r_{H_1}) +$

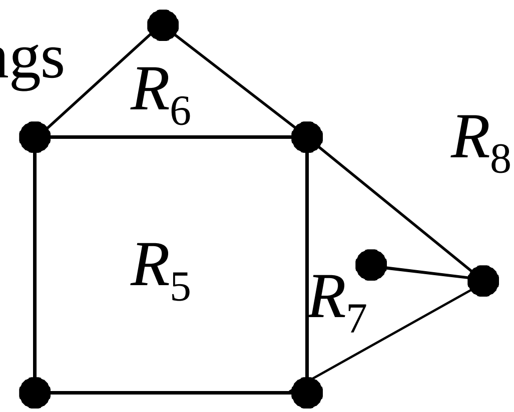
$(v_{H_2} - e_{H_2} + r_{H_2}) - 2 = 2 + 2 - 2 = 2$

Planar Graphs

degree of a region (deg(R)): the number of edges traversed in a shortest closed walk about the boundary of R.

two different embeddings





deg(R_1)=5,deg(R_2)=3
deg(R_3)=3,deg(R_4)=7 → *abghgfda*

deg(R_5)=4,deg(R_6)=3
deg(R_7)=5,deg(R_8)=6

$$\sum_{i=1}^{4} \deg(R_i) = 18 = \sum_{i=5}^{8} \deg(R_i) = 2 \times 9 = 2|E|$$

Planar Graphs

Corollary 1Let $G = (V, E)$ be a loop - free connected planar graph with $|V| = v, |E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v - 6$.

Proof : Since G is a loop - free and is not a multigraph, the boundary of each region (including the infinite region) contains at least three edges. Hence, each region has degree ≥ 3.

Consequently, $2e = 2|E| =$ the sum of the degrees of the r regions determined by G and $2e \geq 3r$. From Euler's theorem,

$$2 = v - e + r \leq v - e + \frac{2e}{3} = v - \frac{e}{3}, \text{ so } 6 \leq 3v - e, \text{ or } e \leq 3v - 6.$$

Only a necessary condition, not sufficient.

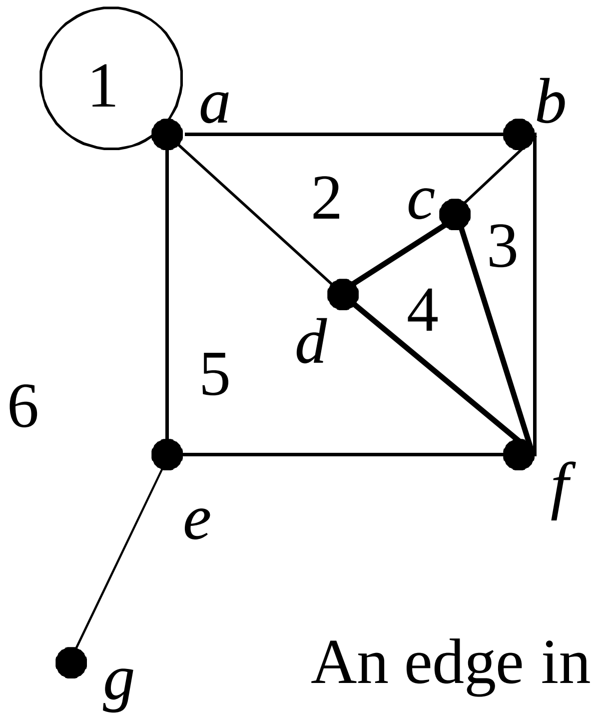
Planar Graphs

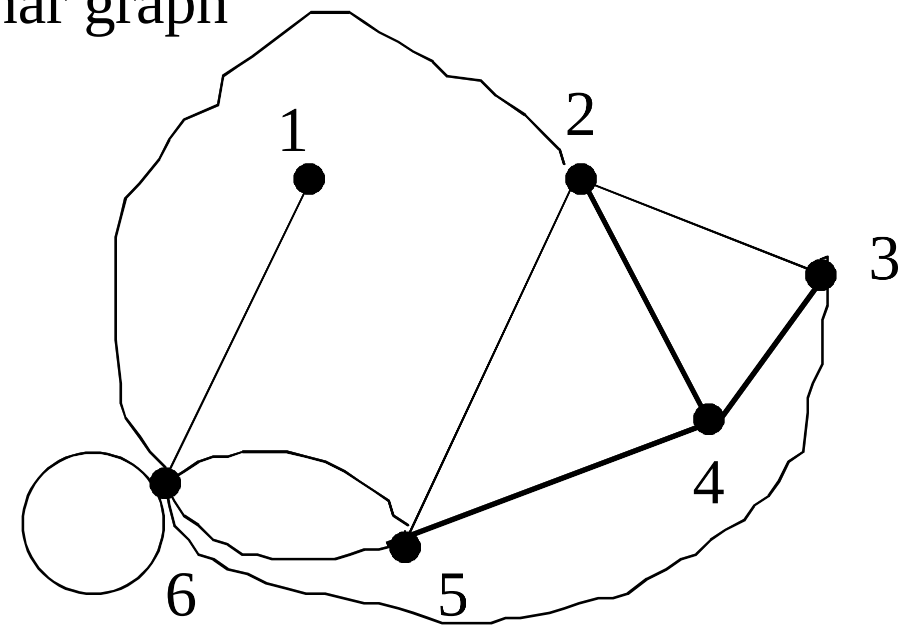
Ex. 3 For K_5, e=10,v=5, 3v-6=9<10=e. Therefore, by Corollary 11.3, K_5 is nonplanar.

Ex. 4 For $K_{3,3}$, each region has at least 4 edges, hence $4r \leq 2e$. If $K_{3,3}$ is planar, $r=e-v+2=9-6+2=5$. So $20=4r \leq 2e=18$, a contradiction.

Planar Graphs

A dual graph of a planar graph



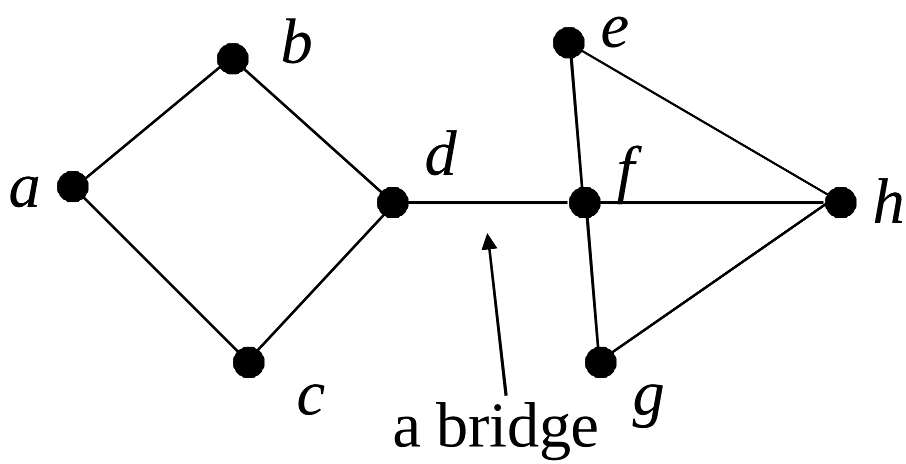


An edge in *G* corresponds with an edge in G^d.

It is possible to have isomorphic graphs with respective duals that are not isomorphic.

cut-set: a subset of edges whose removal increase the number of components

Ex. 5



cut-sets: {(a,b),(a,c)}, {(b,d),(c,d)},{(d,f)},...

For planar graphs, cycles in one graph correspond to cut-sets in a dual graphs and vice versa.

Application & scope of research of Planar Graphs

applications: VLSI routing, plumbing, graph coloring,...

Scope of research : Fast planning through planning graph analysis, Artificial intelligence