DISCRETE STRUCTURE

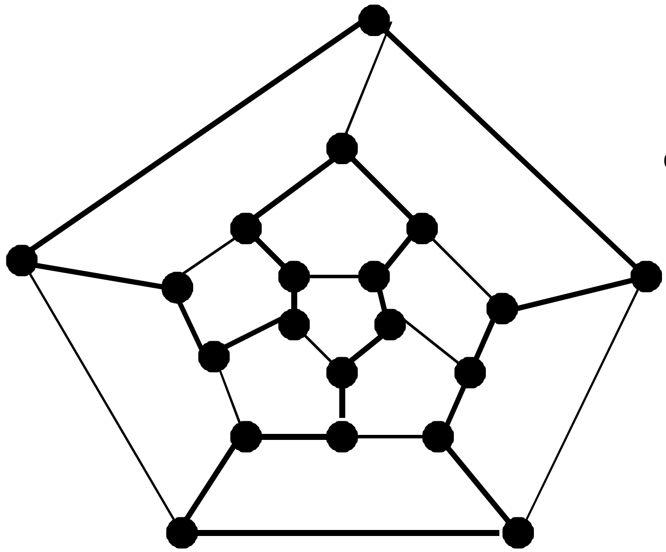
Lecture-26

Hamilton Path & Circuit

Topics covered

- Hamilton path and cycles

Introduction to Hamilton path & circuit

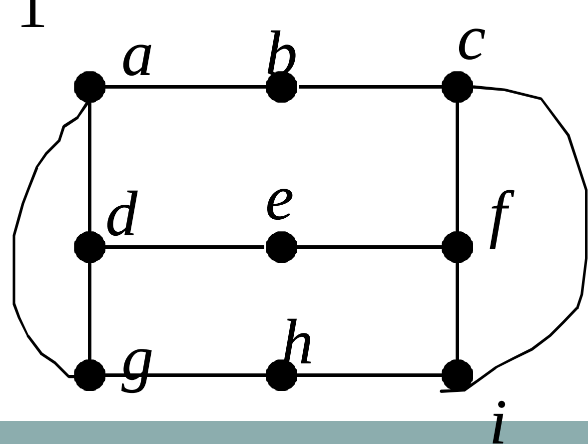


a path or cycle that contain every vertex

Unlike Euler circuit, there is **no known necessary and sufficient condition** for a graph to be Hamiltonian.

an NP-complete problem

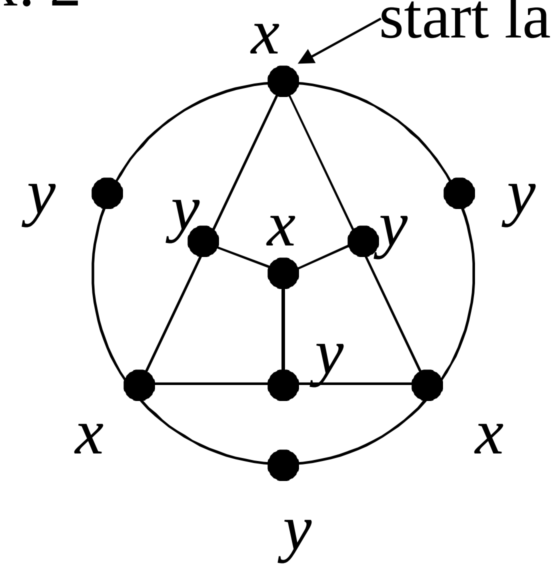
Ex. 1



There is a Hamilton path, but no Hamilton cycle.

Hamilton Paths and Cycles

Ex. 2



4*x*'s and 6*y*'s, since *x* and *y* must interleave in a Hamilton path (or cycle), the graph is not Hamiltonian

The method works only for bipartite graphs.

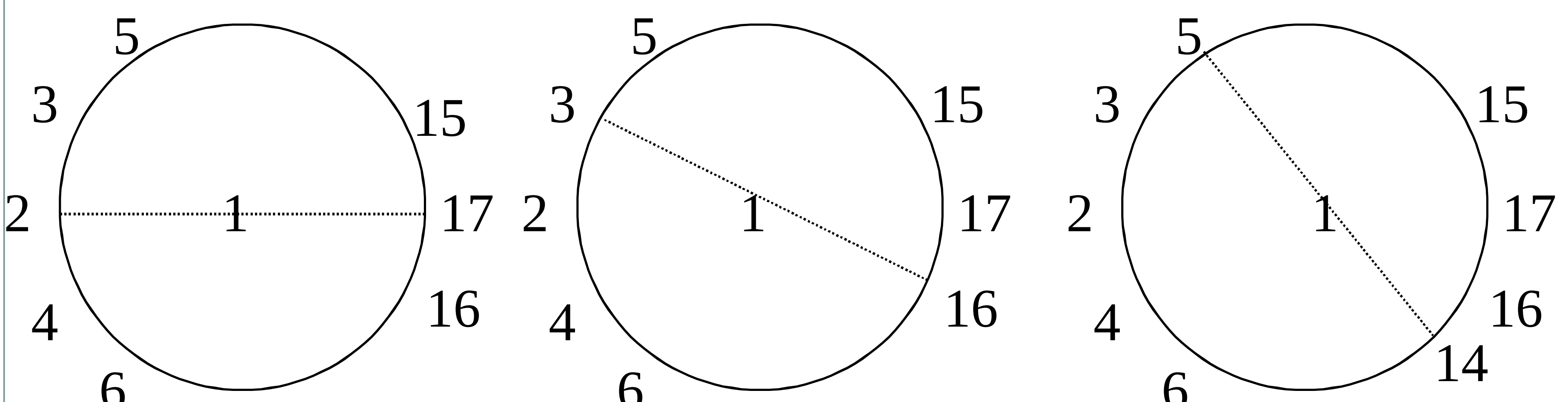
The Hamilton path problem is still NP-complete when restricted to bipartite graphs.

Hamilton Paths and Cycles

Hamilton Paths and Cycles

Ex. 3 17 students sit at a circular table, how many sittings are there such that one has two different neighbors each time?

Consider K_{17}, a Hamilton cycle in K_{17} corresponds to a seating arrangements. Each cycle has 17 edges, so we can have (1/17)17(17-1)/2=8 different sittings.



1,2,3,4,5,6,...,17,1 1,3,5,2,7,4,...,17,14,16,1 1,5,7,3,9,2,...,16,12,14,1

Hamilton Paths and Cycles

Theorem 1 Let K_n^* be a complete directed graph, i.e., K_n^* has n vertices and for any distinct pair x, y of vertices, exactly one of the edges (x, y) or (y, x) is in K_n^*. Such a graph (called a *tournament*) always contains a directed Hamilton path.

Proof : Let $m \geq 2$ with p_m a path containing m - 1 edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)$. If $m = n$, we're finished. If not, let v be a vertex that doesn't appear in p_m.

case 1. $v \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots v_m$

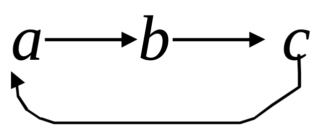
case 2. $v_1 \longrightarrow v_2 \longrightarrow \ldots v_k \longrightarrow v \longrightarrow v_{k+1} \longrightarrow \ldots v_m$

case 3. $v_1 \longrightarrow v_2 \longrightarrow \ldots v_m \longrightarrow v$

Hamilton Paths and Cycles

Ex. 4 In a round-robin tournament each player plays every other player exactly once. We want to somehow rank the players according to the result of the tournament.

not always possible to have a ranking where a player in a certain position has beaten all of the opponents in later positions



but by Theorem 1, **it is possible to list the players such that each has beaten the next player on the list**

Hamilton Paths and Cycles

Theorem 2 Let $G=(V,E)$ be a loop - free graph with $|V|=n\geq 2$. If $\deg(x)+\deg(y)\geq n-1$ for all $x,y\in V$, $x\neq y$, then G has a Hamilton path.

Proof: First prove that G is connected. If not,

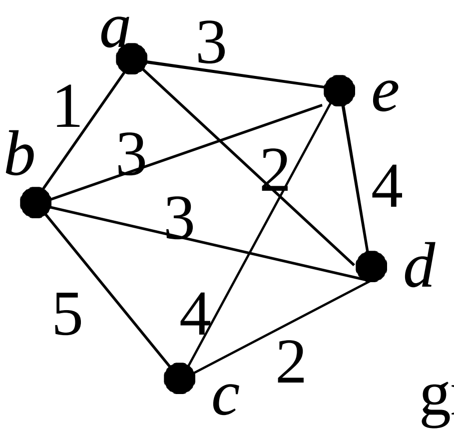
n_1 vertices n_2 vertices

$$\deg(x)+\deg(y)\leq(n_1-1)+(n_2-1)=n_1+n_2-2<n_1+n_2-1$$

a contradiction

Hamilton Paths and Cycles

A related problem: the traveling salesman problem



Find a Hamilton cycle of shortest total distance.

For example, a-b-e-c-d-a with total cost= 1+3+4+2+2=12.

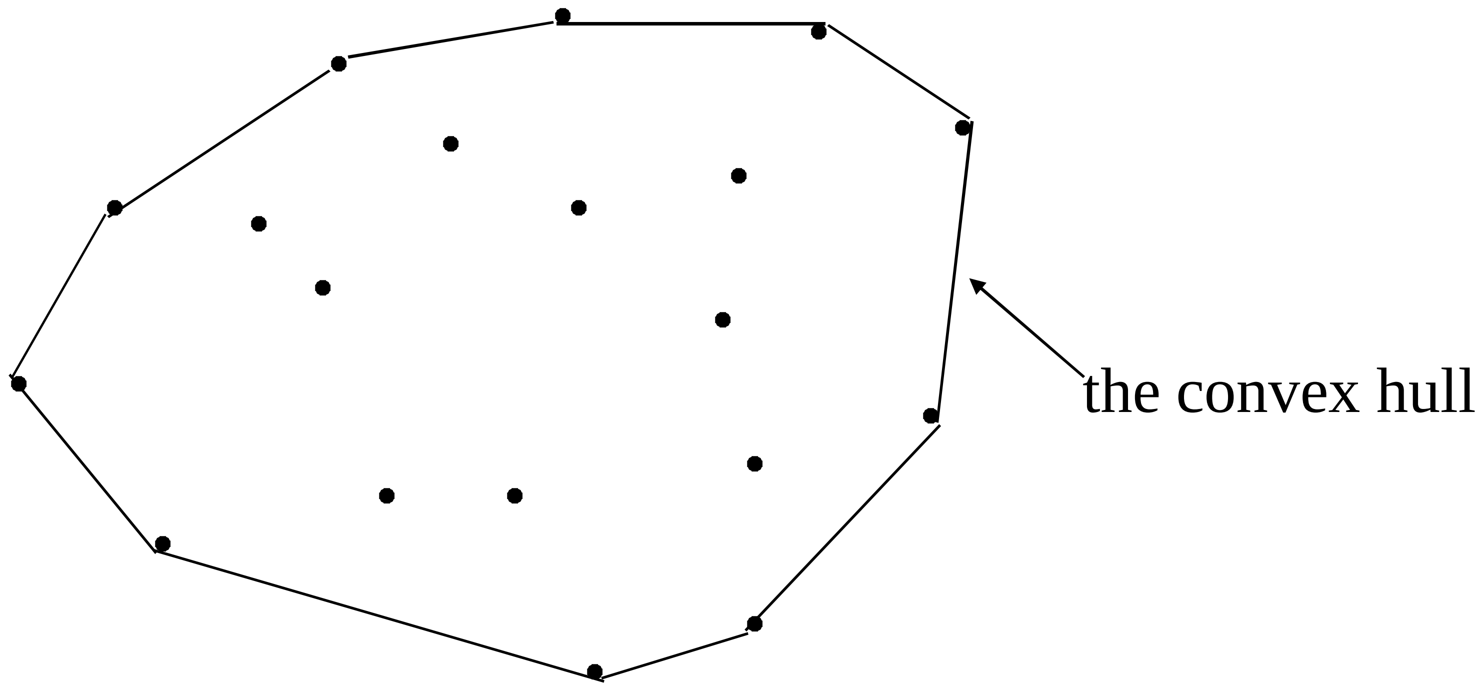
graph problem vs. Euclidean plane problem
(computational geometry)

Certain geometry properties (for example, the triangle inequality) sometimes (but not always) make it simpler.

Hamilton Paths and Cycles

Two famous computational geometry problems.

1. closest pair problem: which two points are nearest
2. convex hull problem



Application

- Hamiltonian cycles in fault random geometric network
- In a network, if Hamiltonian cycles exist, the fault tolerance is better.