DISCRETE STRUCTURE

Lecture-25

Euler path & circuit

Topics covered

- Introduction to Euler path and circuit
- Euler trails

Introduction to Euler path & circuit

An pictorial way to motivate the graph theoretic concepts of Eulerian and Hamiltonian paths and circuits is with two puzzles:

- The pencil drawing problem
- The taxicab problem

Euler Paths and Circuits Definition

DEF: An ***Euler path*** in a graph *G* is a simple path containing every edge in *G*. An ***Euler circuit*** (or ***Euler cycle***) is a cycle which is an Euler path.

NOTE: The definition applies both to undirected as well as directed graphs of all types.

Vertex Degree: Euler Trails and Circuits

degree 1 vertex: pendant vertex

Theorem 1 For simple graphs, $\sum_{v_i \in V} \deg(v_i) = 2|E|$

Corollary 1 The number of vertices of odd degree must be even.

Ex. 1 a *regular* graph: each vertex has the same degree
Is it possible to have a 4-regular graph with 10 edges?

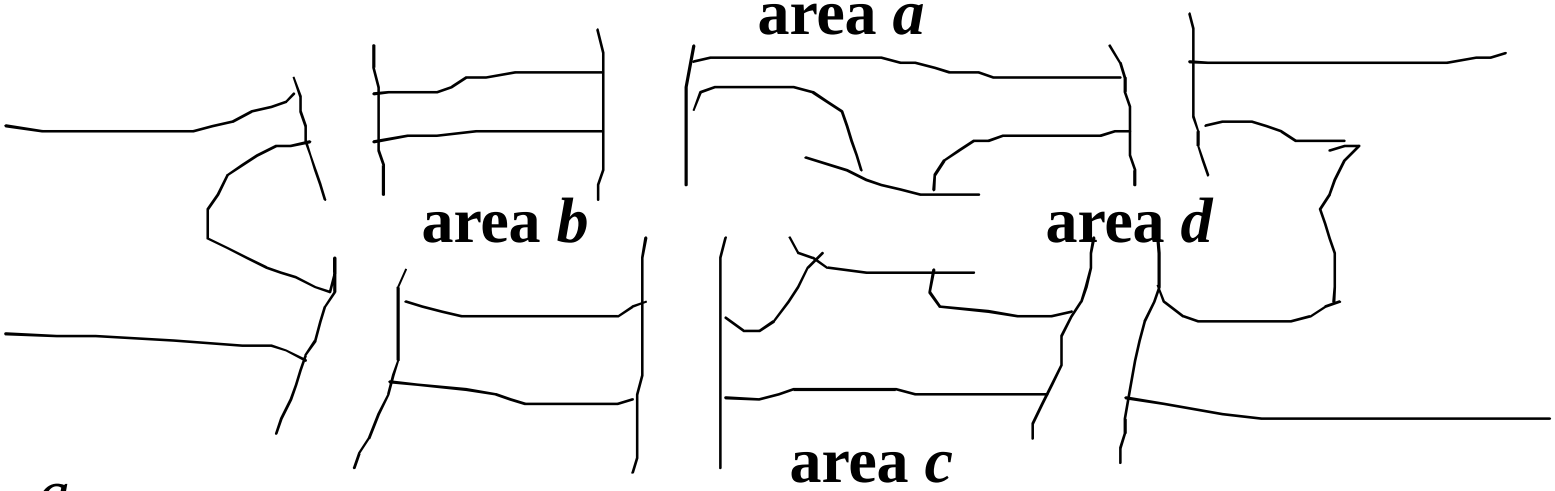
$2|E|=4|V|=20$, $|V|=5 \longrightarrow$ possible (K_5)

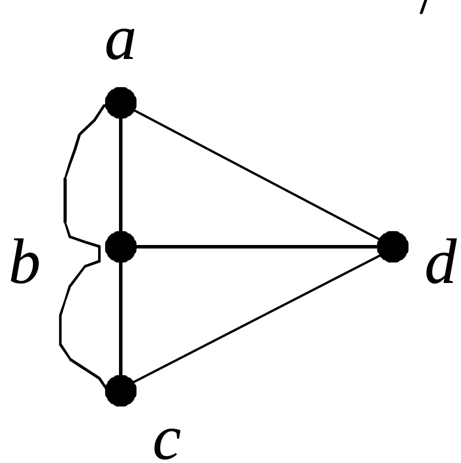
with 15 edges?

$2|E|=4|V|=30 \longrightarrow$ not possible

Vertex Degree: Euler Trails and Circuits

Ex. 2 *The Seven Bridge of Konigsberg*



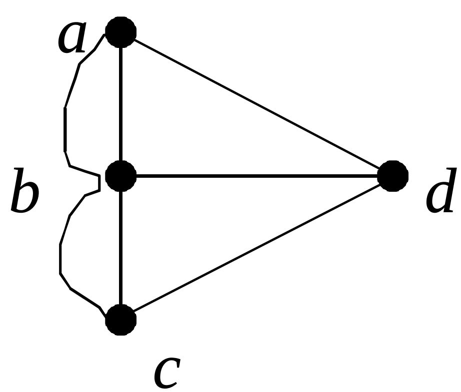


Find a way to walk about the city so as to cross each bridge exactly once and then return to the starting point.

Vertex Degree: Euler Trails and Circuits

Def. 1 Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an *Euler circuit* if there is a circuit in G that t*raverses every edge of the graph exactly once*. If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an *Euler trail*.

Theorem 2 Let $G=(V,E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.



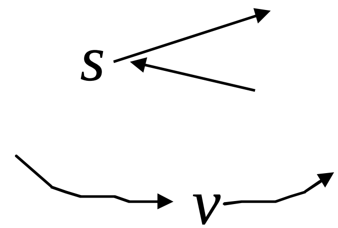
All degrees are odd. Hence no Euler circuit for the Konigsberg bridges problem.

Vertex Degree: Euler Trails and Circuits

proof of Euler circuit theorem:

Euler circuit ⟶ connected and even degree

obvious



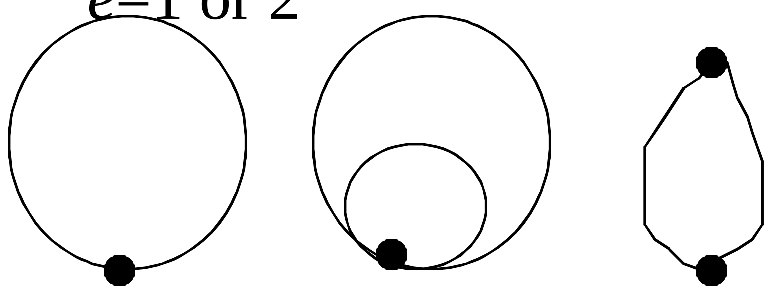
for starting vertex

for other vertices

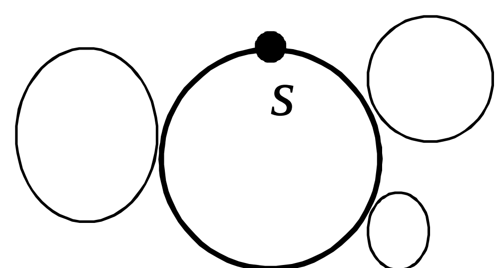
connected and even degree ⟶ Euler circuit

by induction on the number of edges.

e=1 or 2



e=n find any circuit containing s



Vertex Degree: Euler Trails and Circuits

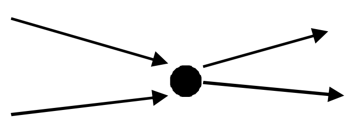
Can you think of an algorithm to construct an Euler circuit?

Corollary 2 An Euler trail exists in *G* if and only if *G* is connected and has exactly two vertices of odd degree.

two odd degree vertices

add an edge

Theorem 2 A directed Euler circuit exists in *G* if and only if *G* is connected and in-degree(*v*)=out-degree(*v*) for all vertices *v*.



one in, one out

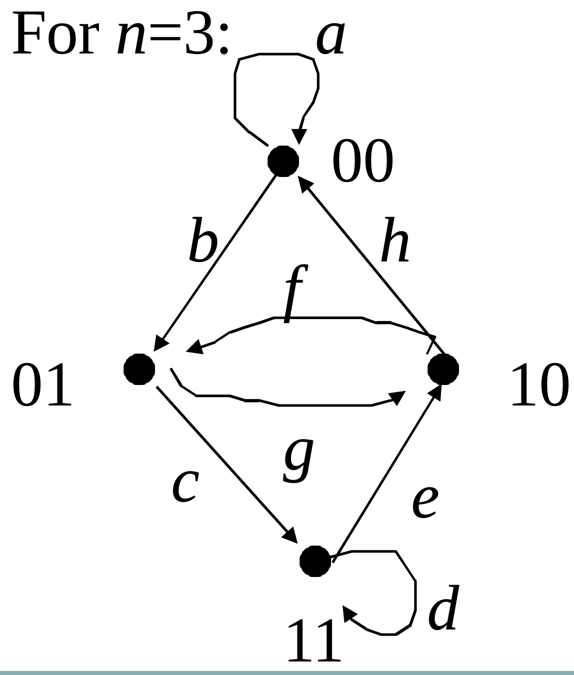
Vertex Degree: Euler Trails and Circuits

Ex. 3 Complete Cycles (DeBruijn Sequences)

If n is a positive integer and $N=2^n$, a cycle of length N of 0's and 1's is called a complete cycle if all possible subsequences of 0's and 1's of length n appear in this cycle.

$n=1$	01,
$n=2$	0011,
$n=3$	00010111,00011101
$n=4$	16 complete cycles
In general	$2^{2^{n-1}-n}$

For $n=3$:



vertex set={00,01,10,11}

a directed edge from x_1x_2 to $x_2\ x_3$

Find an Euler circuit:

abcdefgh ⟶ 00111010

abgfcdeh ⟶ 00101110

Application & Scope of research of Euler circuit

Application :Euler circuits and paths are also useful to painters, garbage collectors, airplane pilots and all world navigators, like

Scope of research : Euler circuits with no monochromatic transitions in edge colored graph

Euler Formula

Euler proved that for any connected graph
f = no. of faces
e = no. of edges
v = no. of vertices
Then the formula hold
f = e – v -2