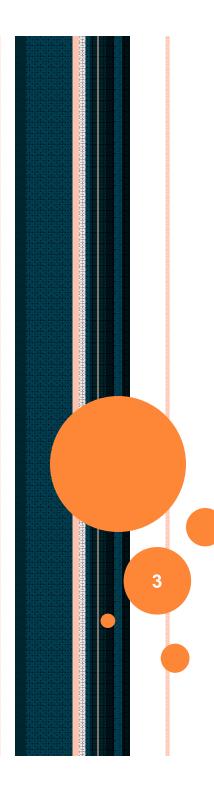


DISCRETE STRUCTURE



COSETS & LANGRE'S

2



TOPICS COVERED

COSETS

LAGRANGE'S THEOREM

- Let (G, ·) be a group with subgroup H. For a, b ∈ G, we say that a is *congruent to b modulo H*, and write a ≡ b mod H if and only if ab⁻¹ ∈ H.
- Proposition 2.1. 1. The relation a ≡ b mod H is an equivalence relation on G. The equivalence class containing a can be written in the form Ha = {ha|h ∈ H}, and it is called a right coset of H in G. The element a is called a representative of

the coset Ha.

• Example 1. Find the right cosets of A₃ in S₃.

Solution. One coset is the subgroup itself $A_3 = \{(1), (123), (132)\}$. Take any element not in the subgroup, say (12). Then another coset is $A_3(12) = \{(12), (123), (12), (132), (12)\} = \{(12), (13), (23)\}$.Since the right cosets form a partition of S_3 and the two cosets above contain all the elements of S_3 , it follows that these are the only two cosets. In fact, $A_3 = A_3(123) = A_3(132)$ and $A_3(12) = A_3(13) = A_3(23)$.

- Example 2.1.2. Find the right cosets of $H = \{e, g^4, g^8\}$ in $C_{12} = \{e, g, g^2, ..., g^{11}\}$.
- Solution. H itself is one coset. Another is Hg = {g, g⁵, g⁹}. These two cosets have not exhausted all the elements of C₁₂, so pick an element, say g², which is not in H or Hg. A third coset is Hg² = {g², g⁶, g¹⁰} and a fourth is Hg³ = {g³, g⁷, g¹¹}.

Since $C_{12} = H \cup Hg \cup Hg^2 \cup Hg^3$, these are all the cosets

COSETS & LAGARANGE

• 2.2.Theorem of Lagrange

- As the examples above suggest, every coset contains the same number of elements.
- Lemma 2.2.1. There is a bijection between any two right cosets of H in G.
- *Proof.* Let Ha be a right coset of H in G. We produce a bijection between Ha and H, from which it follows that there is a bijection between any two right cosets.

Define ψ :H \rightarrow Ha by $\psi(h) = ha$. Then ψ is clearly surjective. Now suppose that $\psi(h_1) = \psi(h_2)$, so that $h_1a = h_2a$. Multiplying each side by a^{-1} on the right, we obtain $h_1a = h_2$. Hence ψ is a bijection.

LAGARNGE

- **Theorem 2.** Lagrange's Theorem. If G is a finite group and H is a subgroup of G, then |H| divides |G|.
- *Proof.* The right cosets of H in G form a partition of G, so G can be written as a disjoint union
- $G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_k$ for a finite set of elements $a_1, a_2, \ldots, a_k \in G$.
- By Lemma 2.2.1, the number of elements in each coset is |H|. Hence, counting all the elements in the disjoint union above, we see that |G| = k|H|. Therefore, |H| divides |G|.

COSETS & LAGRANGE

- If H is a subgroup of G, the number of distinct right cosets of H in G is called the *index of H in G* and is written |G : H|. The following is a direct consequence of the proof of Lagrange's theorem.
- **Corollary 2.2.3**. If G is a finite group with subgroup *H*, then

|G : H| = |G|/|H|.

• **Corollary 2.2.4**. If a is an element of a finite group *G*, then the order of a divides the order of *G*.

Let G be a group with subgroup H. The *right cosets* of H in G are equivalence classes under the relation a ≡ b mod H, defined by ab⁻¹ ∈ H. We can also define the relation L on G so that aLb if and only if b⁻¹a ∈ H. This relation, L, is an equivalence relation, and the equivalence class containing a is the *left coset* aH = {ah|h ∈ H}. As the following example shows, the left coset of an element does not necessarily equal the right coset.

- Example 2.3.1. Find the left and right cosets of $H = A_3$ and $K = {(1), (12)}$ in S₃.
- Solution. We calculated the right cosets of $H = A_3$ in Example 2.1.1. Right Cosets

$$H = \{(1), (123), (132)\}; H(12) = \{(12), (13), (23)\}$$

Left Cosets

 $H = \{(1), (123), (132); (12)H = \{(12), (23), (13)\}$

In this case, the left and right cosets of *H* are the same.

However, the left and right cosets of K are not all the same.
Right Cosets

 $K = \{(1), (12)\}$; $K(13) = \{(13), (132)\}$; $K(23) = \{(23), (123)\}$ Left Cosets

 $K = \{(1), (12)\}; (23)K = \{(23), (132)\}; (13)K = \{(13), (123)\}$

APPLICATION & SCOPE OF RESEARCH

- Cosets of Q in R are used in the construction of Vital Sets, a type of nonmeasurable set.
- Cosets are important in computational group theory
- Linear error correcting code
- Automata theory
- Scope of research : Coding theory