DISCRETE STRUCTURE

LECTURE-21

COSETS & LANGRE'S

Topics covered

Cosets

Lagrange's theorem

COSETS

- Let (*G, ·*) be a group with subgroup H. For a, b ∈ G, we say that a is ***congruent to b modulo H*****,** and write **a ≡ b mod H** if and only if $ab^{-1} \in H$.
- **Proposition 2.1.** ***1.****The relation a ≡ b mod H is an equivalence relation on G. The equivalence class containing a can be written in the form Ha = {ha|h ∈ H}, and it is called a right coset of H in G. The element a is called a representative of*

  the coset Ha.

COSETS

- **Example 1. Find the right cosets of A_3 *in* S_3.**

Solution. One coset is the subgroup itself $A_3 = \{(1), (123), (132)\}$. Take any element not in the subgroup, say (12). Then another coset is $A_3(12) = \{(12), (123)\ (12), (132)\ (12)\} = \{(12), (13), (23)\}$. Since the right cosets form a partition of S_3 and the two cosets above contain all the elements of S_3, it follows that these are the only two cosets.

In fact, $A_3 = A_3(123) = A_3(132)$ *and* $A_3(12) = A_3(13) = A_3(23)$.

COSETS

- **Example 2.1.2. Find the right cosets of *$H = \{e, g^4, g^8\}$ in $C_{12} = \{e, g, g^2, \ldots, g^{11}\}$.***
- *Solution.* H itself is one coset. Another is $Hg = \{g, g^5, g^9\}$. These two cosets have not exhausted all the elements of C_{12}, so pick an element, say g^2, which is not in H or Hg. A third coset is $Hg^2 = \{g^2, g^6, g^{10}\}$ and a fourth is $Hg^3 = \{g^3, g^7, g^{11}\}$.

  Since $C_{12} = H \cup Hg \cup Hg^2 \cup Hg^3$, these are all the cosets

- **2.2.Theorem of Lagrange**
- As the examples above suggest, every coset contains the same number of elements.
- **Lemma 2.2.1.** *There is a bijection between any two right cosets of H in G.*

Proof. Let Ha be a right coset of H in G. We produce a bijection between Ha and H, from which it follows that there is a bijection between any two right cosets.

Define $\psi: H \to Ha$ by $\psi(h) = ha$. Then ψ is clearly surjective. Now suppose that $\psi(h_1) = \psi(h_2)$, so that $h_1 a = h_2 a$. Multiplying each side by a^{-1} on the right, we obtain $h_1 = h_2$. Hence ψ is a bijection.

LAGARNGE

- **Theorem 2.** *Lagrange's Theorem. If G is a finite group and H is a subgroup of G, then |H| divides |G|.*

Proof. The right cosets of H in G form a partition of G, so G can be written as a disjoint union

$G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_k$ for a finite set of elements $a_1, a_2, \ldots, a_k \in G$.

By Lemma 2.2.1, the number of elements in each coset is |H|. Hence, counting all the elements in the disjoint union above, we see that |G| = k|H|. Therefore, |H| divides |G|.

Cosets & Lagrange

- If H is a subgroup of G, the number of distinct right cosets of H in G is called the *index of H in G* and is written |G : H|. The following is a direct consequence of the proof of Lagrange's theorem.
- **Corollary 2.2.3**. *If G is a finite group with subgroup H, then*

  $$|G : H| = |G|/|H|.$$
- **Corollary 2.2.4**. *If a is an element of a finite group G, then the order of a divides the order of G*.

COSETS

- Let G be a group with subgroup H. The *right cosets* of H in G are equivalence classes under the relation $a \equiv b \bmod H$, defined by $ab^{-1} \in H$. We can also define the relation L on G so that aLb if and only if $b^{-1}a \in H$. This relation, L, is an equivalence relation, and the equivalence class containing a is the *left coset* $aH = \{ah | h \in H\}$. As the following example shows, the left coset of an element does not necessarily equal the right coset.

COSETS

- **Example 2.3.1. Find the left and right cosets of *$H = A_3$ and $K = \{(1), (12)\}$ in S_3.***
- *Solution. We calculated the right cosets of $H = A_3$ in Example 2.1.1.*

  Right Cosets

  $H = \{(1), (123), (132)\}$; $H(12) = \{(12), (13), (23)\}$

  Left Cosets

  $H = \{(1), (123), (132\}$; $(12)H = \{(12), (23), (13)\}$

  In this case, the left and right cosets of *H are the same.*
- However, the left and right cosets of *K are not all the same.*

  Right Cosets

  $K = \{(1), (12)\}$; $K(13) = \{(13), (132)\}$; $K(23) = \{(23), (123)\}$

  Left Cosets

  $K = \{(1), (12)\}$;$(23)K = \{(23), (132)\}$; $(13)K = \{(13), (123)\}$

APPLICATION & SCOPE OF RESEARCH

- Cosets of **Q** in **R** are used in the construction of Vital Sets, a type of nonmeasurable set.
- Cosets are important in computational group theory
- Linear error correcting cod**e**
- Automata theory
- Scope of research : Coding ttheory