DISCRETE STRUCTURE

LECTUIRE-16

Recurrence Relations

TOPICS COVERED

- Recurrence Relations
- Modeling with Recurrence Relations

Recurrence Relations

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Recurrence Relations

- In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

- Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

- If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relations

- **Example:**

Consider the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

- Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?
- For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.

- Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Recurrence Relations

- Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?
- For $n \geq 2$ we see that $2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$.
- Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Modeling with Recurrence Relations

- **Example:**
- Someone deposits $10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?
- **Solution:**
- Let P_n denote the amount in the account after n years.
- How can we determine P_n on the basis of P_{n-1}?

Modeling with Recurrence Relations

- We can derive the following **recurrence relation**:
- $P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.
- The initial condition is $P_0 = 10{,}000$.
- Then we have:
- $P_1 = 1.05P_0$
- $P_2 = 1.05P_1 = (1.05)^2P_0$
- $P_3 = 1.05P_2 = (1.05)^3P_0$
- …
- $P_n = 1.05P_{n-1} = (1.05)^nP_0$

- We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

- Let us use this formula to find P_{30} under the
- initial condition $P_0 = 10,000$:

- $P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$

- After 30 years, the account contains \$43,219.42.

Solving Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:
- $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$,
- Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions
- $a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

Solving Recurrence Relations

- **Examples:**
- The recurrence relation $P_n = (1.05)P_{n-1}$
- is a linear homogeneous recurrence relation of **degree one**.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$
- is a linear homogeneous recurrence relation of **degree two**.
- The recurrence relation $a_n = a_{n-5}$
- is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

- Basically, when solving such recurrence relations, we try to find solutions of the form $a_n = r^n$, where r is a constant.
- $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ if and only if
- $r^n = c_1r^{n-1} + c_2r^{n-2} + \dots + c_kr^{n-k}$.
- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1r^{k-1} - c_2r^{k-2} - \dots - c_{k-1}r - c_k = 0$
- This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

- **Example:** What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

- **Solution:** The characteristic equation of the recurrence relation is $r^2 – r – 2 = 0$.
- Its roots are $r = 2$ and $r = -1$.
- Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:
- $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Solving Recurrence Relations

- Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that
- $a_0 = 2 = \alpha_1 + \alpha_2$
- $a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

- Solving these two equations gives us $\alpha_1 = 3$ and $\alpha_2 = -1$.

- Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with
- $a_n = 3 \cdot 2^n – (-1)^n$.

Solving Recurrence Relations

- $a_n = r^n$ is a solution of the linear homogeneous recurrence relation
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$
- if and only if
- $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.
- Divide this equation by r^{n-k} and subtract the right-hand side from the left:
- $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$
- This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

- **Example:** Give an explicit formula for the Fibonacci numbers.
- **Solution:** The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.
- The characteristic equation is $r^2 - r - 1 = 0$.
- Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations

- Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

Solving Recurrence Relations

- The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations

- But what happens if the characteristic equation has only one root?
- How can we then match our equation with the initial conditions a_0 and a_1 ?
- **Theorem:** Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.

A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Solving Recurrence Relations

- **Example:** What is the solution of the recurrence relation $a_n = 6a_{n-1} – 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?
- **Solution:** The only root of $r^2 – 6r + 9 = 0$ is $r_0 = 3$. Hence, the solution to the recurrence relation is
- $a_n = \alpha_1 3^n + \alpha_2 n3^n$ for some constants α_1 and α_2.
- To match the initial condition, we need
- $a_0 = 1 = \alpha_1$
  $a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$
- Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.
- Consequently, the overall solution is given by
- $a_n = 3^n + n3^n$.