DISCRETE STRUCTURE

LECTURE-15

Linear Recurrences with Constant Coefficients

TOPICS COVERED

- Linear Recurrences with Constant Coefficients
- Homogeneous Linear Recurrences

Linear Recurrences with Constant Coefficients

Previous method generalizes to solving "*linear recurrence relations with constant coefficients*":

DEF: A recurrence relation is said to be ***linear*** if a_n is a linear combination of the previous terms plus a function of *n*. I.e. no squares, cubes or other complicated function of the previous a_i can occur. If in addition all the coefficients are constants then the recurrence relation is said to have ***constant coefficients***.

Linear Recurrences with Constant Coefficients

Q: Which of the following are linear with constant coefficients?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$
3. $a_n = a_{n-1}^2$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: YES
3. $a_n = a_{n-1}^2$: NO. Squaring is not a linear operation. Similarly $a_n = a_{n-1}a_{n-2}$ and $a_n = \cos(a_{n-2})$ are non-linear.

Homogeneous Linear Recurrences

To solve such recurrences we must first know how to solve an easier type of recurrence relation:

DEF: A linear recurrence relation is said to be ***homogeneous*** if it is a linear combination of the previous terms of the recurrence *without* an additional function of n.

Q: Which of the following are homogeneous?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: No. There's an extra term $f(n) = 2^{n-3}$

Homogeneous Linear Recurrences with Const. Coeff.'s

- The 3-step process used for the next example, Fibonacci recurrence, works well for general homogeneous linear recurrence relations with constant coefficients. There are a few instances where some modification is necessary.

In class notes are useful

Solving Fibonacci

Recipe solution has 3 basic steps:

1) Assume solution of the form $a_n = r^n$
2) Find all possible r's that seem to make this work. Call these[1] r_1 and r_2. Modify assumed solution to ***general solution*** $a_n = Ar_1^n + Br_2^n$ where A,B are constants.
3) Use initial conditions to find A,B and obtain specific solution.

Solving Fibonacci

1) Assume exponential solution of the form $a_n = r^n$:

Plug this into $a_n = a_{n-1} + a_{n-2}$:

$$r^n = r^{n-1} + r^{n-2}$$

Notice that all three terms have a common r^{n-2} factor, so divide this out:

$$r^n / r^{n-2} = (r^{n-1} + r^{n-2}) / r^{n-2} \Rightarrow r^2 = r + 1$$

This equation is called the ***characteristic equation*** of the recurrence relation.

Solving Fibonacci

2) Find all possible r's that solve characteristic

$$r^2 = r + 1$$

Call these r_1 and r_2.[1] General solution is $a_n = Ar_1^n + Br_2^n$ where A,B are constants.

Quadratic formula[2] gives:

$$r = (1 \pm \sqrt{5})/2$$

So $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$

General solution:

$$a_n = A\,[(1+\sqrt{5})/2]^n + B\,[(1-\sqrt{5})/2]^n$$

Solving Fibonacci

3) Use initial conditions $a_0 = 0$, $a_1 = 1$ to find A,B and obtain specific solution.

$0=a_0 = A\,[(1+\sqrt{5})/2]^0 + B\,[(1-\sqrt{5})/2]^0 = A + B$

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 = A(1+\sqrt{5})/2 + B\,(1-\sqrt{5})/2 = (A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2nd:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

(CHECK IT!) $$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Homogeneous-Complications

Repeating roots in characteristic equation. Repeating roots imply that don’t learn anything new from second root, so may not have enough information to solve formula with given initial conditions. We’ll see how to deal with this on next slide.

Complication: Repeating Roots

EG: Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$

Find characteristic equation by plugging in $a_n = r^n$:

$$r^2 - 2r + 1 = 0$$

Since $r^2 - 2r + 1 = (r - 1)^2$ the root $r = 1$ repeats.

SOLUTION: Multiply second solution by n so general solution looks like:

$$a_n = Ar_1^n + Bnr_1^n$$

Complication: Repeating Roots

Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$

General solution: $a_n = A1^n + Bn1^n = A + Bn$

Plug into initial conditions

$1 = a_0 = A + B \cdot 0 \cdot 1^0 = A$

$2 = a_0 = A \cdot 1^1 + B \cdot 1 \cdot 1^1 = A + B$

Plugging first equation $A = 1$ into second:

$2 = 1 + B$ implies $B = 1$.

Final answer: $a_n = 1 + n$

(CHECK IT!)