Discrete Structure

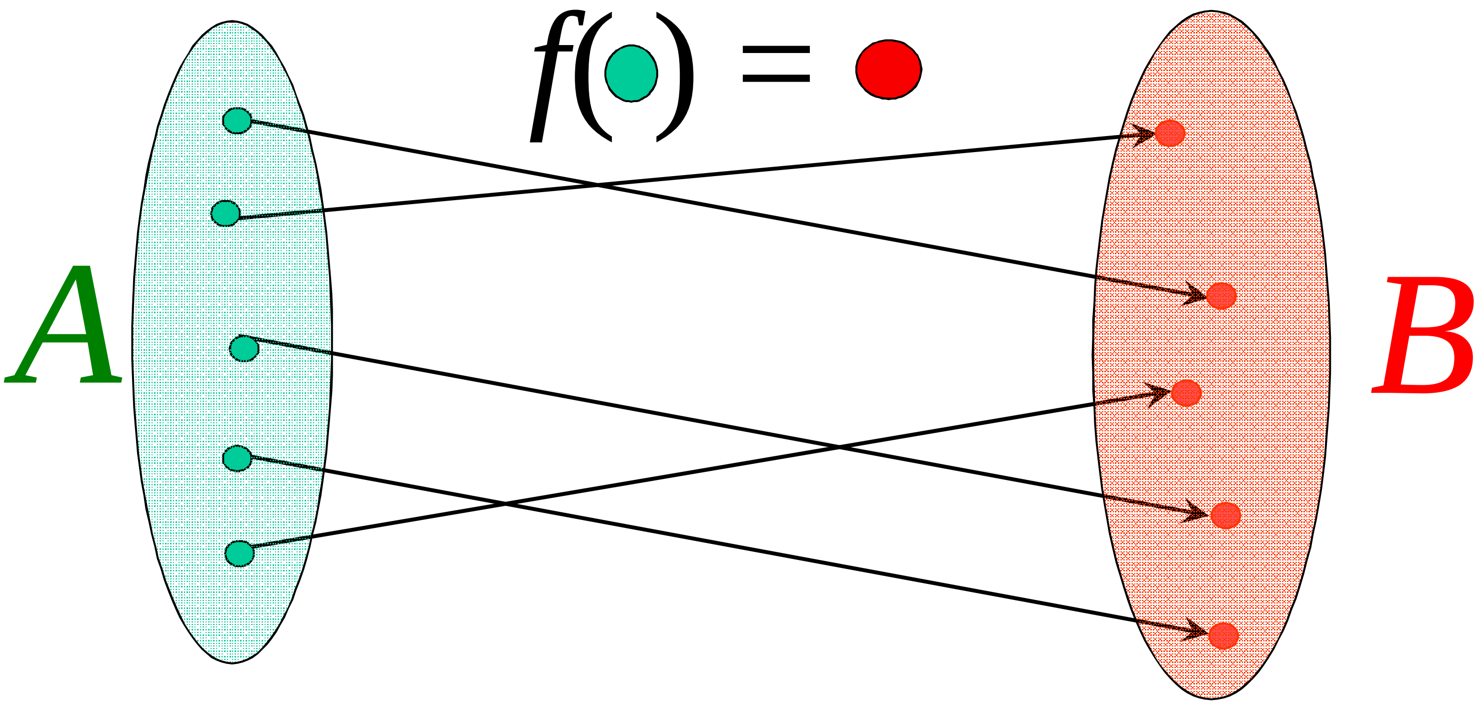
Lecture-4

Function and its types

Topics Covered

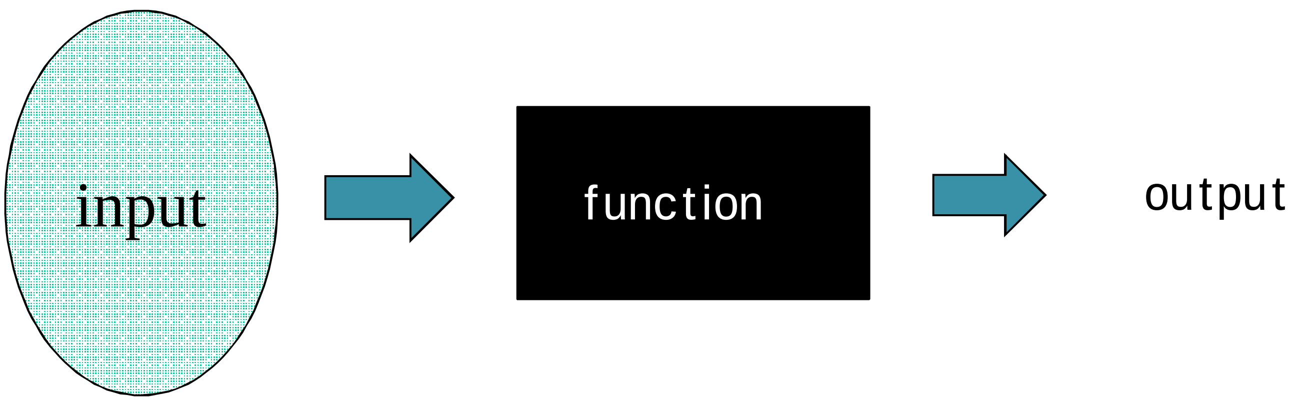
- Functions
- Pigeonhole principle
- Application of functions

Functions, Pigeonhole Principle



Functions

Informally, we are given an “input set”,
and a function gives us an output for each possible input.



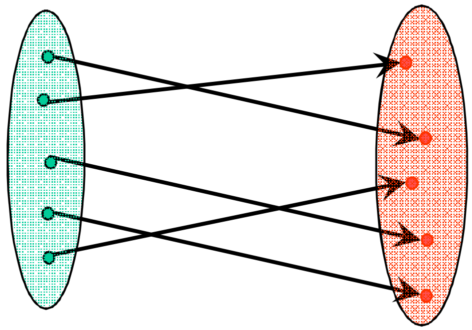
The important point is that there is only one output for each input.

We say a function f “maps” the element of an input set A
to the elements of an output set B.

Functions

More formally, we write $f : A \rightarrow B$

to represent that *f* is a function from set *A* to set *B,* which associates an element $f(a) \in B$ with an element $a \in A.$



The *domain (input)* of *f* is A.

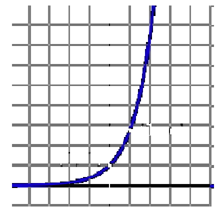
The *codomain (output)* of *f* is B.

Definition: For every input there is exactly one output.

Note: the input set can be the same as the output set, e.g. both are integers.

Examples of Functions

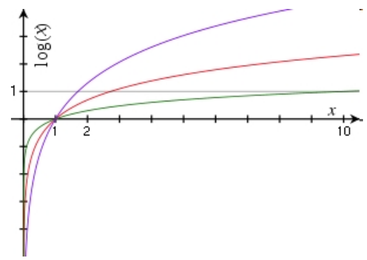
$$f(x) = e^x$$



domain = R

codomain = $R^{>0}$

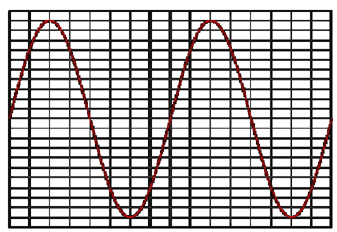
$$f(x) = \log(x)$$



domain = $R^{>0}$

codomain = R

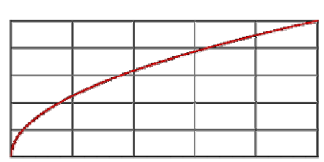
$$f(x) = \sin(x)$$



domain = R

codomain = [-1,1]

$$f(x) = \sqrt{x}$$



domain = $R^{>=0}$

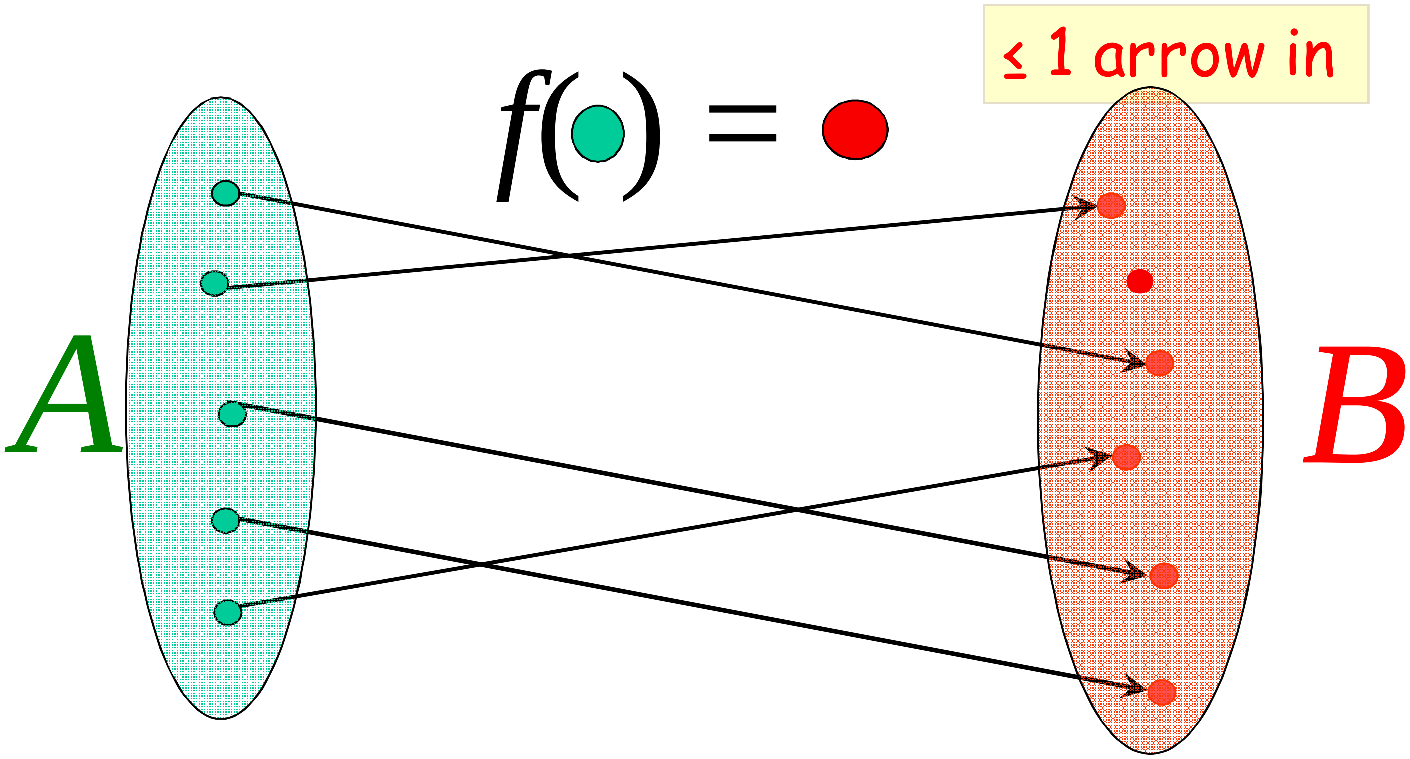
codomain = $R^{>=0}$

Examples of Functions

f(S) = |S|	domain = the set of all finite sets codomain = non-negative integers
f(string) = length(string)	domain = the set of all finite strings codomain = non-negative integers
f(student-name) = student-ID	not a function, since one input could have more than one output
f(x) = is-prime(x)	domain = positive integers codomain = {T,F}

Injections

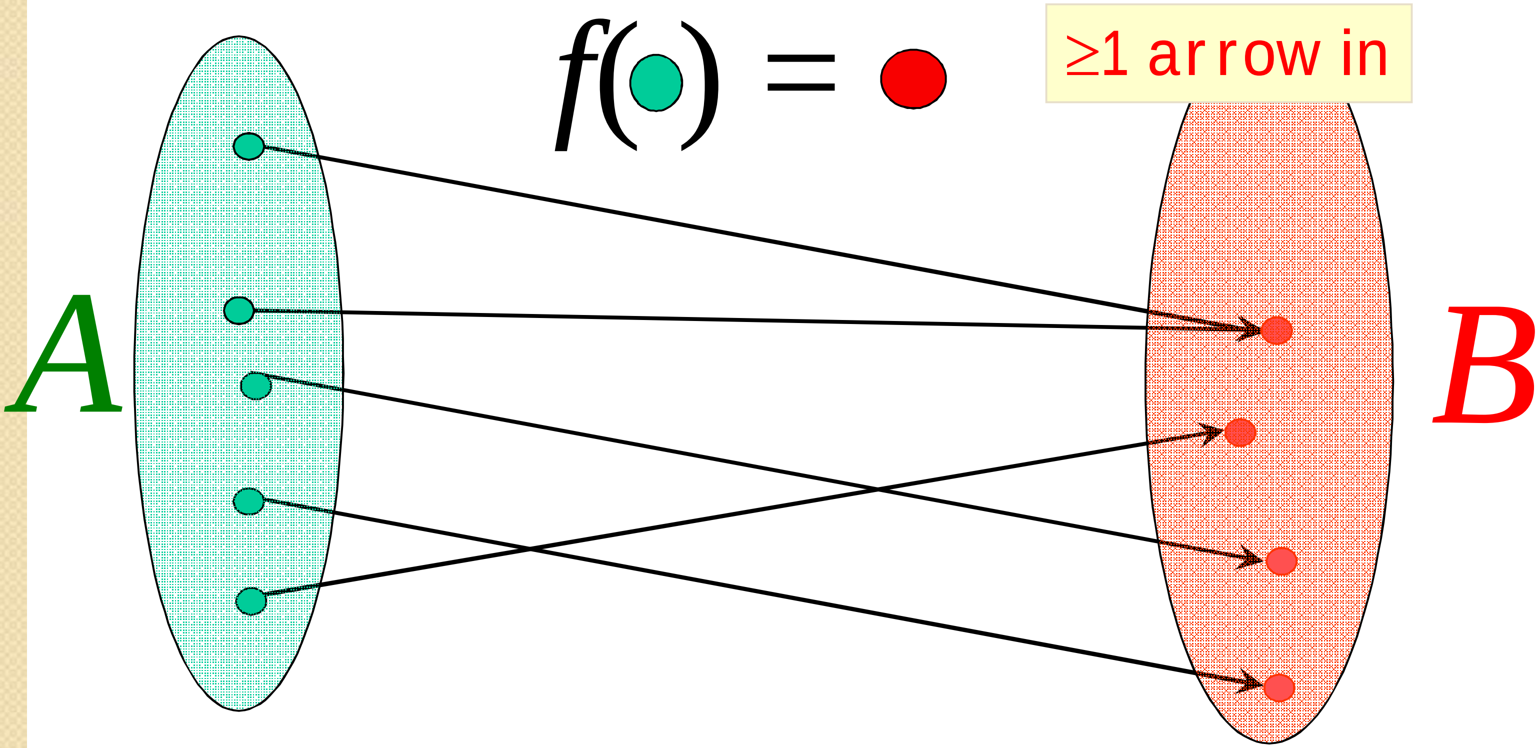
$f : A \to B$ is an *injection* if no two inputs have the same output.



$|A| \leq |B|$

Surjections

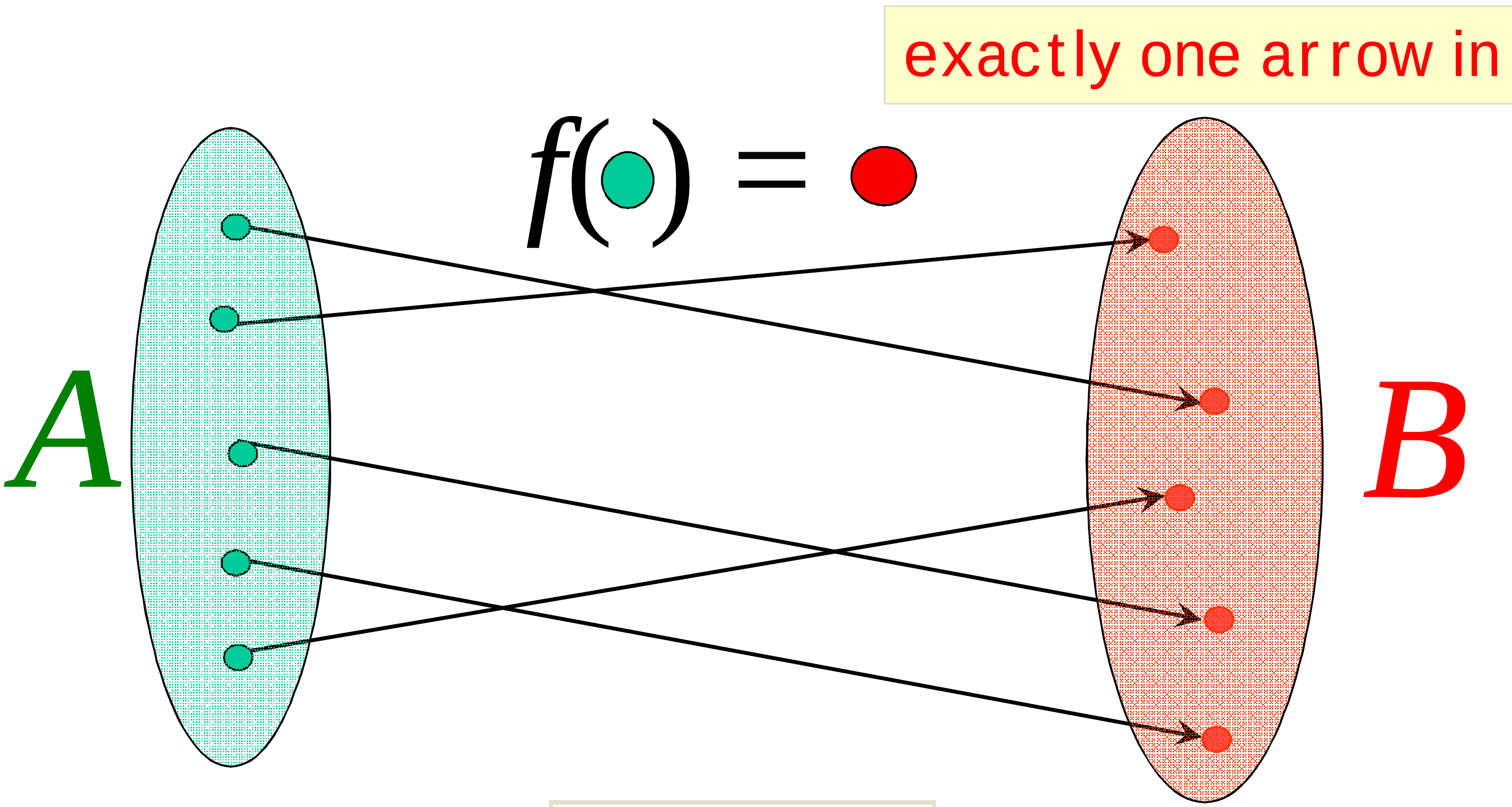
$f : A \to B$ is a *surjection* if every output is possible.



$|A| \geq |B|$

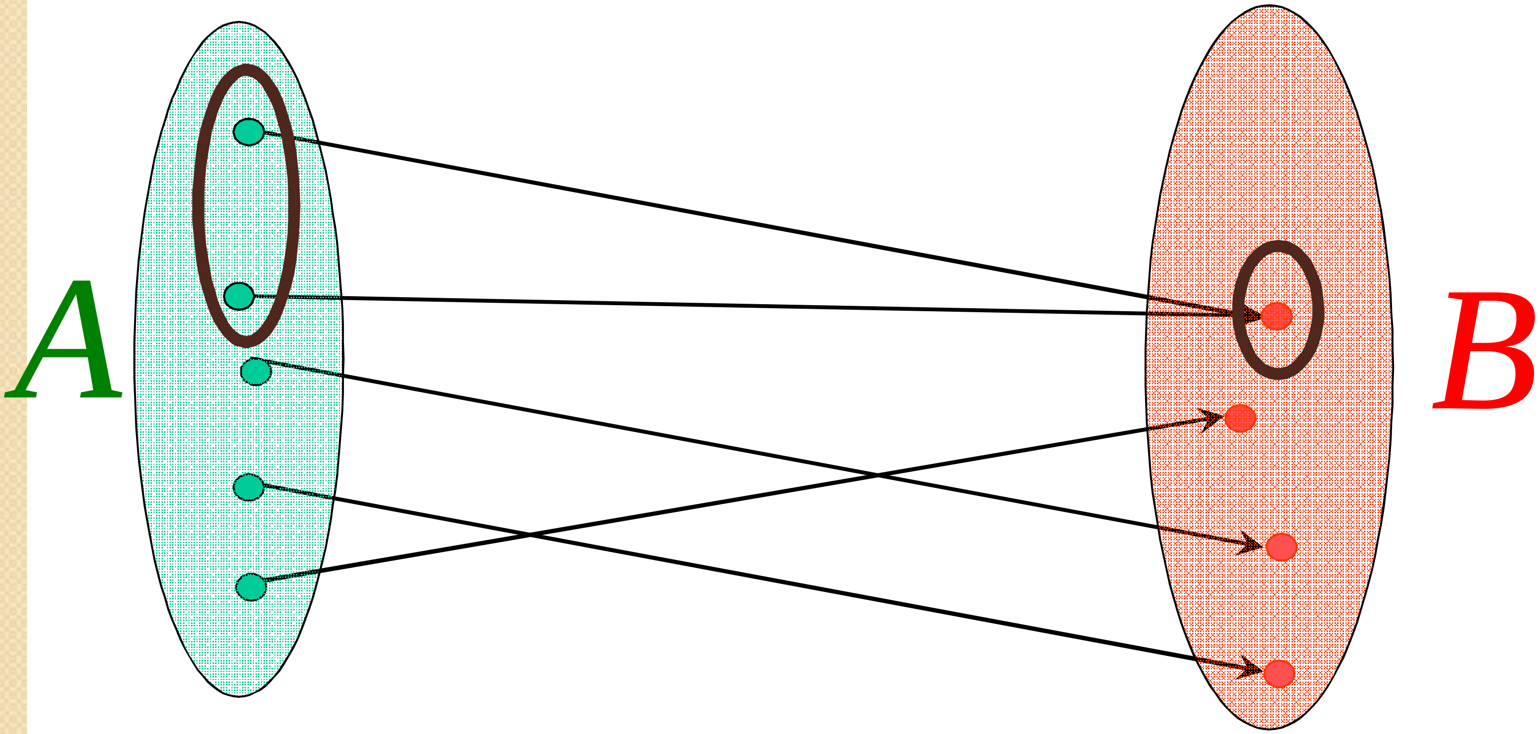
Bijections

$f : A \to B$ is a *bijection* if it is surjection and injection.



$|A| = |B|$

Inverse Sets



Given an element y in B, the inverse set of y := $f^{-1}(y) = \{x \text{ in } A \mid f(x) = y\}$.

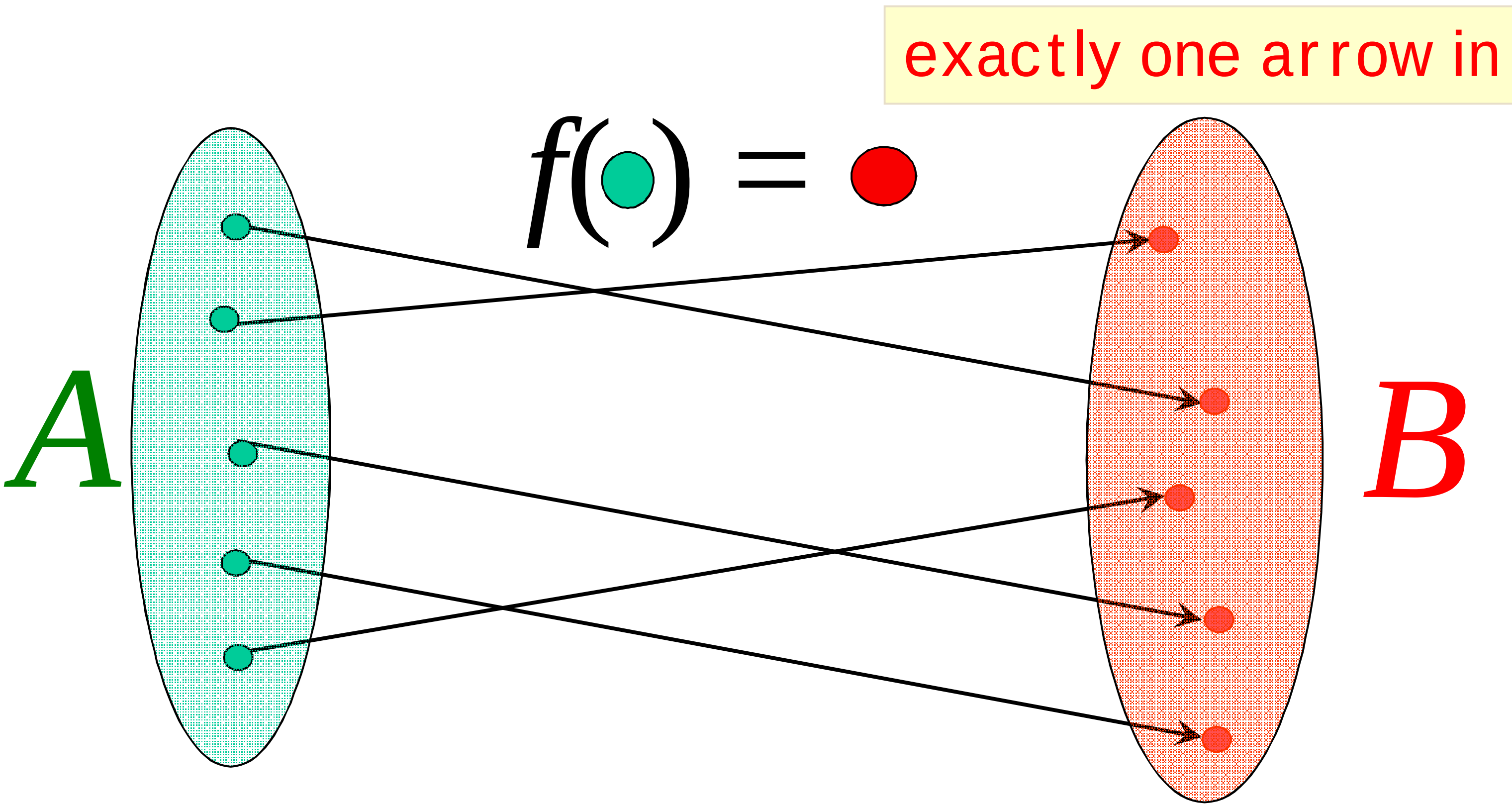
In words, this is the set of inputs that are mapped to y.

More generally, for a subset Y of B,

the inverse set of Y := $f^{-1}(Y) = \{x \text{ in } A \mid f(x) \text{ in } Y\}$.

Inverse Function

Informally, an inverse function f^{-1} is to "undo" the operation of function f.



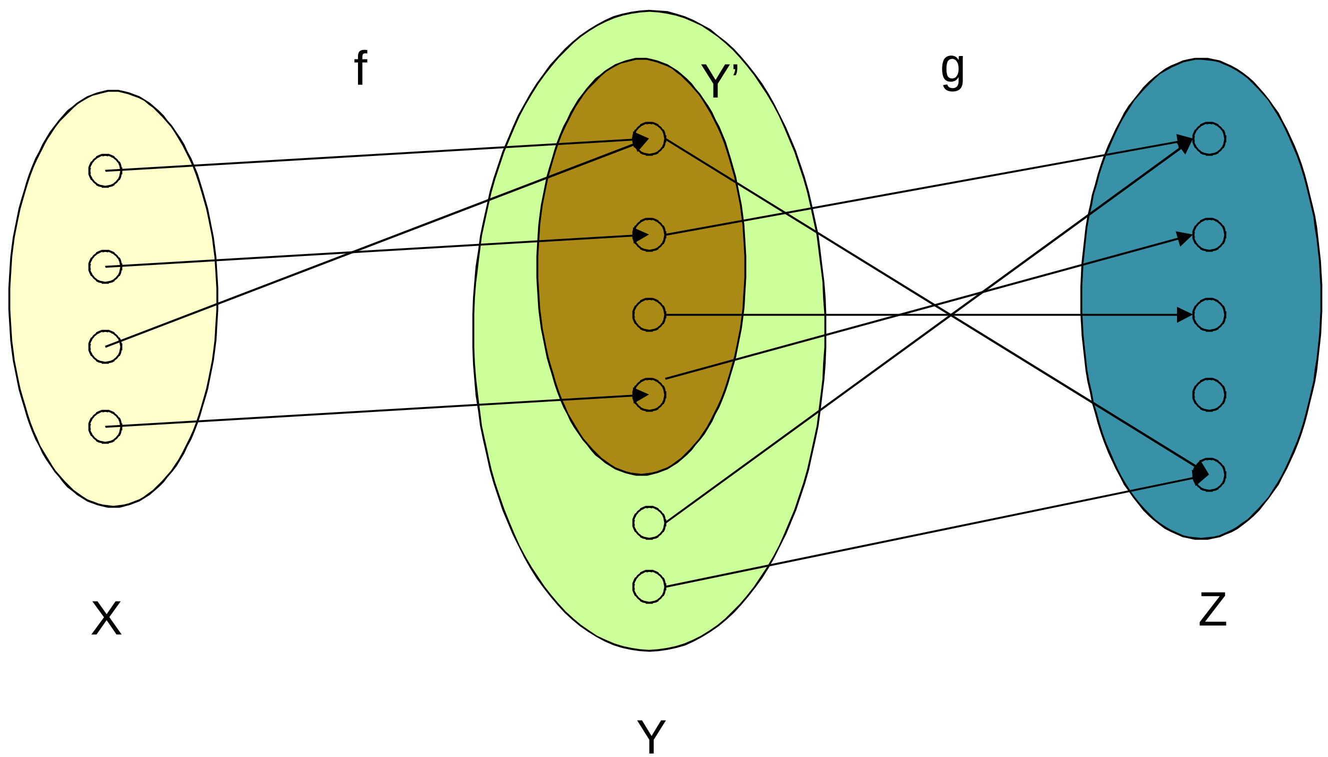
There is an inverse function f^{-1} for f if and only if f is a bijection.

Composition of Functions

Two functions f:X->Y', g:Y->Z so that Y' is a subset of Y,

then the composition of f and g is the function $g \circ f$: X->Z, where

$$g \circ f(x) = g(f(x)).$$

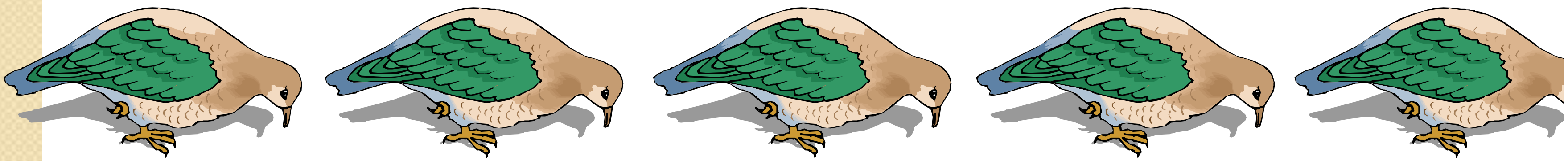


Assignment

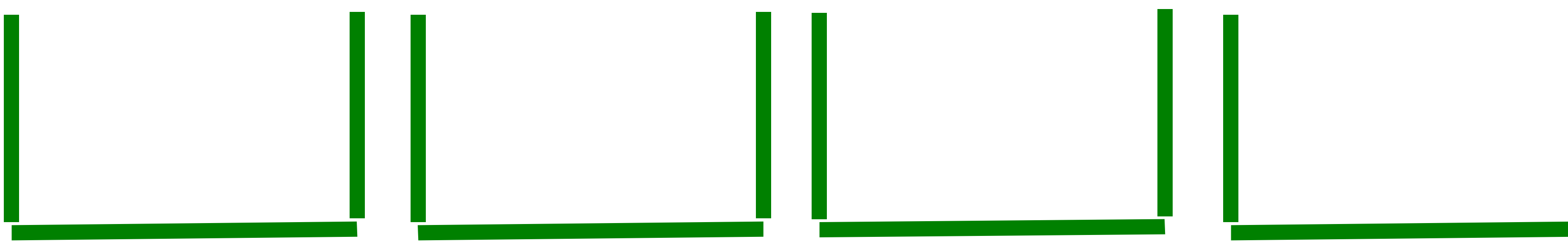
Function f	Function g	$g \circ f$ injective?	$g \circ f$ surjective?	$g \circ f$ bijective?
f:X->Y f surjective	g:Y->Z g injective			
f:X->Y f surjective	g:Y->Z g surjective			
f:X->Y f injective	g:Y->Z g surjective			
f:X->Y f bijective	g:Y->Z g bijective			
f:X->Y	f^{-1}:Y->X			

Pigeonhole Principle

If more pigeons

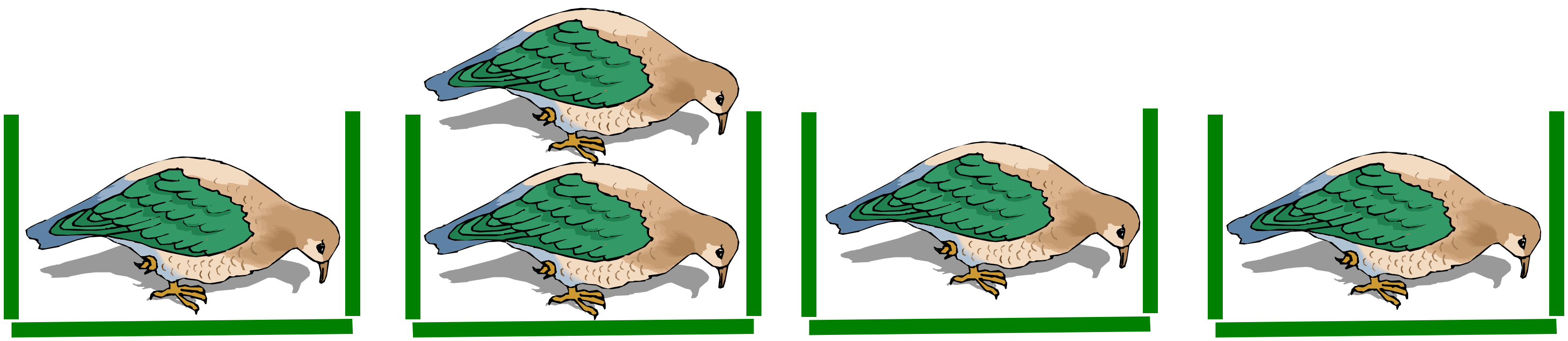


than pigeonholes,



Pigeonhole Principle

then some hole must have at least two pigeons!



Pigeonhole principle

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that are mapped to the same element in the codomain.)

Example 1

Question: Let A = {1,2,3,4,5,6,7,8}

If five distinct integers are selected from A, must a pair of integers have a sum of 9?

Consider the pairs {1,8}, {2,7}, {3,6}, {4,5}.

The sum of each pair is equal to 9.

If we choose 5 numbers from the set A,

then by the pigeonhole principle,

both elements of some pair will be chosen,

and their sum is equal to 9.

Example 2

Question: In a party of n people, is it always true that there are two people shaking hands with the same number of people?

Everyone can shake hand with 0 to n-1 people, and there are n people, and so it does not seem that it must be the case, but think about it carefully:

Case 1: if there is a person who does not shake hand with others, then any person can shake hands with at most n-2 people, and so everyone shakes hand with 0 to n-2 people, and so the answer is “yes” by the pigeonhole principle.

Case 2: if everyone shakes hand with at least one person, then any person shakes hand with 1 to n-1 people, and so the answer is “yes” by the pigeonhole principle.

Application of POSET: Birthday Paradox

In a group of 367 people, there must be two people having the same birthday.

Suppose n <= 365, what is the probability that in a random set of n people, some pair of them will have the same birthday?

We can think of it as picking n random numbers from 1 to 365 without repetition.

There are 365^n ways of picking n numbers from 1 to 365.

There are $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1)$ ways of picking n numbers from 1 to 365 without repetition.

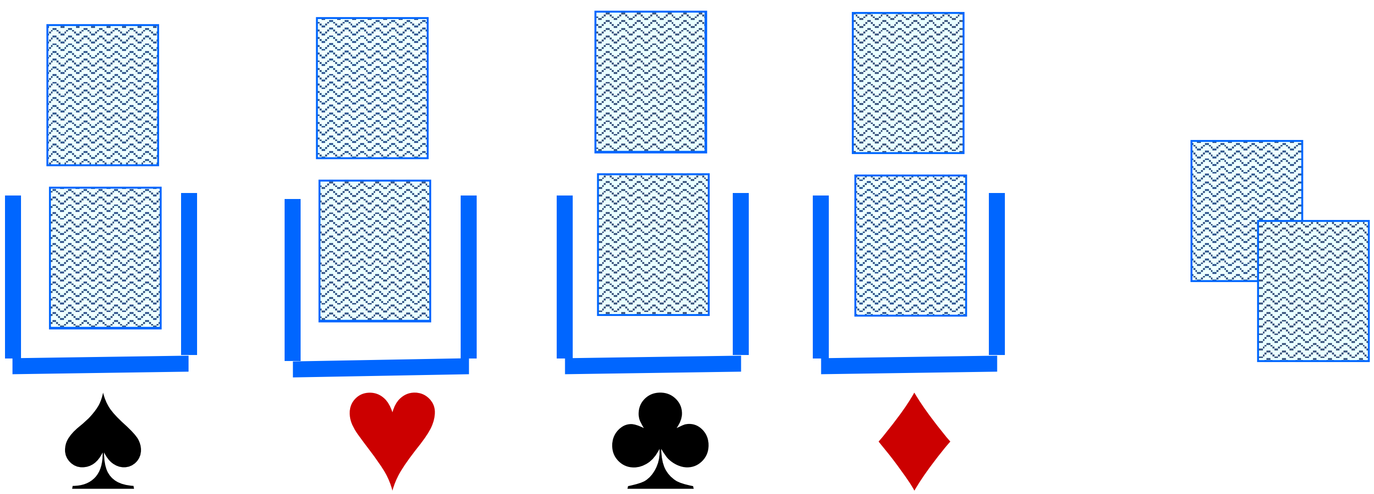
So the probability that no pairs have the same birthday is equal to $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1) / 365^n$

This is smaller than 50% for 23 people, smaller than 1% for 57 people.

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle

If n pigeons and h holes, then some hole has at least $\left\lceil \frac{n}{h} \right\rceil$ pigeons.



Cannot have < 3 cards in every hole.

Application of Function: Subset Sum

We are asking whether two subsets must have the same sum.

We can ask whether the opposite can hold: whether all subsets have different sums.

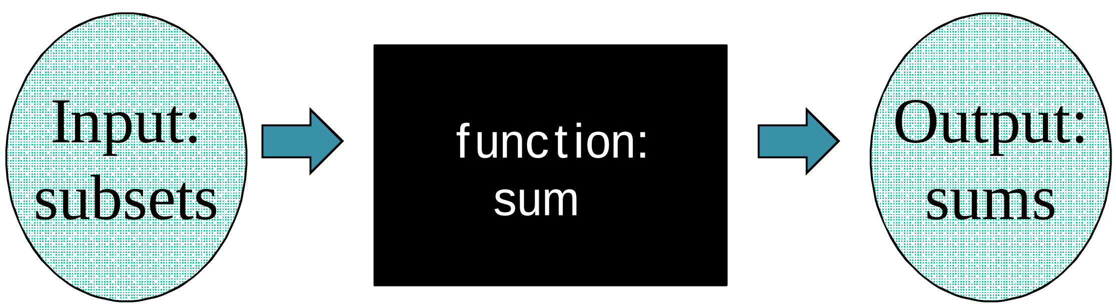
We will show that it is not possible for all subsets to have different sums.

The strategy is by counting.

We will count how many possible different subsets are there, say the answer is X.

And we will also count how many possible different sums are there, say the ans is Y.

If X > Y, then by the pigeonhole principle, there are more inputs than outputs and thus it is not possible for all subsets to have different sums.



Subset Sum

Let A be the set of the 90 numbers, each with at most 25 digits.
So the total sum of the 90 numbers is at most 90×10^{25}.

Let X be the set of all subsets of the 90 numbers. **(pigeons)**

Let Y be the set of integers from 0 to 90×10^{25}. **(pigeonholes)**

Let f:X->Y be a function mapping each subset of A into its sum.

If we could show that |X| > |Y|, then by the pigeonhole principle,
the function f must map two elements in X into the same element in Y.
This means that there are two subsets with the same sum.

Exercises

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x)=\sin(x)$	Real	Real			
$f(x)=2^x$	Real	Positive real			
$f(x)=x^2$	Real	Non-negative real			
Reverse string	Bit strings of length n	Bit strings of length n			

Whether a function is injective, surjective, bijective depends on its domain (i.e. input) and the codomain (i.e. output).

Application & Scope of research

Relational database