DISCRETE STRUCTURE

LECTURE-8

Partial ordering relation & lattice

TOPICS COVERED

Partially Ordered Set (POSET)

Comparable/Incomparable

Totally Ordered, Chains

Well-Ordered Set

Hasse Diagrams

Latices

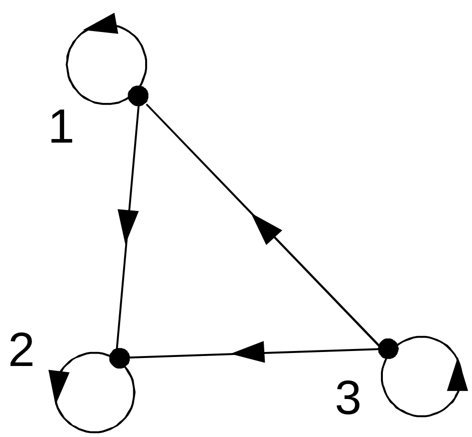
Introduction to Partially Ordered Set (POSET)

A relation R on a set S is called a *partial ordering* or *partial order* if it is *reflexive*, *antisymmetric*, and *transitive*. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R)

Example (1)

Let S = {1, 2, 3} and

let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}



$\preccurlyeq$

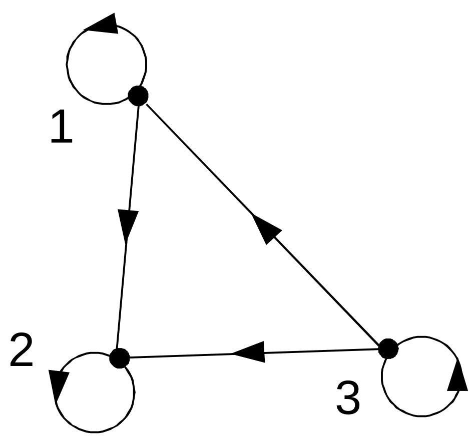
In a poset the notation $a \preccurlyeq b$ denotes that $(a,b) \in R$

This notation is used because the "*less than or equal to*" relation is a paradigm for a partial ordering. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the "less than or equals" relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$

Example

Let S = {1, 2, 3} and

let R = {(1,1), (2,2), (3,3), (1, 2), (3,1), (3,2)}



$2 \preccurlyeq 2$ $3 \prec 2$

Example (2)

Consider the set of real numbers and the "less than or equal to" relation. $(R, \leq)$

Example (3)

Consider the power set of $\{a, b, c\}$ and the subset relation. $(P(\{a,b,c\}), \subseteq\ )$

Comparable/Incomparable

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Example

Consider the power set of {a, b, c} and the subset relation. ($P(\{a,b,c\}), \subseteq$)

$\{a,c\} \not\subseteq \{a,b\}$ and $\{a,b\} \not\subseteq \{a,c\}$

So, {a,c} and {a,b} are *incomparable*

Totally Ordered, Chains

If $(S, \prec)$ is a poset and every two elements of S are comparable, S is called *totally ordered* or *linearly ordered* set, and $\prec$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Example

Consider the set of real numbers and the "less than or equal to" relation. $(R, \leq\ )$

Well-Ordered Set

$(S, \prec)$ is a *well-ordered set* if it is a poset such that $\prec$ is a total ordering and such that every nonempty subset of S has a *least element*.

Example: Consider the ordered pairs of positive integers, $Z^+ \times Z^+$ where

$$(a_1, a_2) \preccurlyeq (b_1, b_2) \text{ if } a_1 < b_1, \text{ or if } a_1 = b_1 \text{ and } a_2 \leq b_2$$

Hasse Diagrams

Given any partial order relation defined on a finite set, it is possible to draw the directed graph so that all of these properties are satisfied.

This makes it possible to associate a somewhat simpler graph, called a *Hasse diagram*, with a partial order relation defined on a finite set.

Hasse Diagrams (continued)

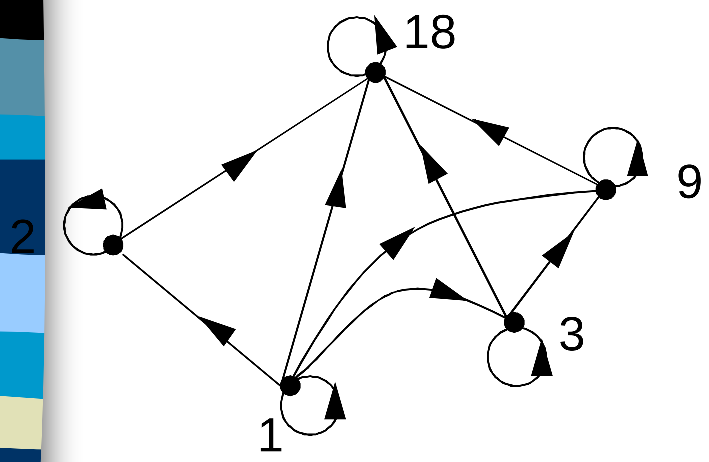
Start with a directed graph of the relation in which all arrows point upward. Then eliminate:

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example

Let A = {1, 2, 3, 9, 19} and consider the “divides” relation on A:

For all $a, b \in A,\ a \mid b \Leftrightarrow b = ka$ for some integer k.

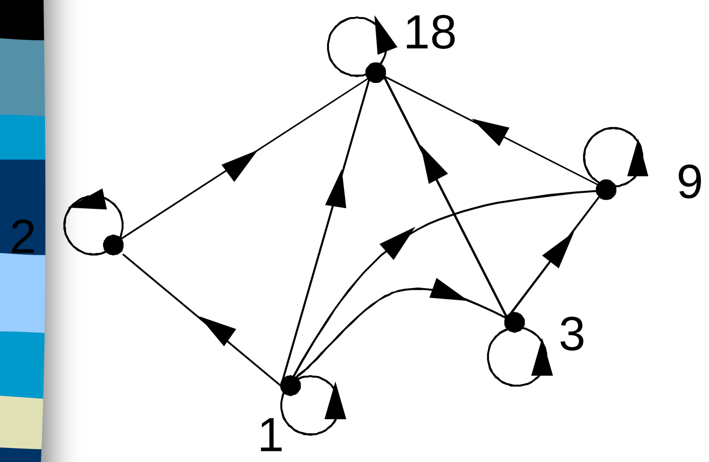


Example

Eliminate the loops at all the vertices.

Eliminate all arrows whose existence is implied by the transitive property.

Eliminate the direction indicators on the arrows.



Maximal and Minimal Elements

a is a *maximal* in the poset (S, $\preccurlyeq$) if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called *minimal* if it is not greater than any element of the poset. That is, a is *minimal* if there is no element $b \in S$ such that $b \prec a$.

It is possible to have multiple minimals and maximals.

Greatest Element
Least Element

a is the *greatest element* in the poset (S,$\preccurlyeq$) if $b \prec a$ for all $b \in S$. Similarly, an element of a poset is called the *least element* if it is less than all other elements in the poset That is, a is *least element* if there is no $b \in S$ such that $b \prec a$.

Upper bound, Lower bound

Sometimes it is possible to find an element that is greater than all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Likewise, there may be an element less than all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Least Upper Bound, Greatest Lower Bound

The element x is called the *least upper bound* of the subset A if x is an upper bound that is less than every other upper bound of A.

The element y is called the *greatest lower bound* of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A.

Lattices

A partially ordered set in which *every pair* of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

Lattice

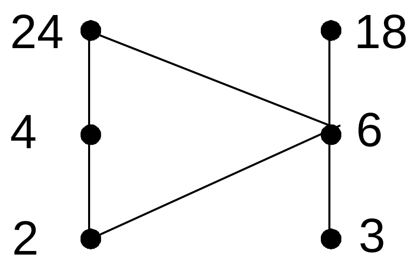
Theorem:
Let <L, ≤> be a lattice. If x*y (x+y) denotes the glb (lub) for {x,y}, then the following holds.
For any a, b, c∈ L,
(i) a*a=a (i') a+a=a (Idempotent)
(ii) a*b=b*a (ii') a+b=b+a (Commutative)
(iii) (a*b)*c= a*(b*c) (iii') (a+b)+c= a+(b+c) (Associative)
(iv) a*(a+b)=a (iv') a+(a*b)=a (Absorption

Topological Sorting

A total ordering $\preccurlyeq$ is said to be compatible with the partial ordering R if $a \preccurlyeq b$ whenever $a\ R\ b$. Constructing a total ordering from a partial ordering is called *topological sorting*.

Example

Consider the set $A = \{2, 3, 4, 6, 18, 24\}$ ordered by the "divides" relation. The Hasse diagram follows:



The ordinary "less than or equal to" relation $\leq$ on this set is a topological sorting for it since for positive integers a and b, if $a|b$ then $a \leq b$.

Application & Scope of Research of POSET

Application

- Assemble an Automobile
- Decision Algorithm

Scope of research

- Expressiveness & complexity in underspecified semantic
- Assisted Calculation proof & proof checking based on POSET