DISCRETE STRUCTURE

LECTURE-3

Introduction to
Equivalence Relations

Introduction

- Certain combinations of relation properties are very useful
  - We won’t have a chance to see many applications in this course
- In this set we will study equivalence relations
  - A relation that is reflexive, symmetric and transitive

Outline

- What is an equivalence relation
- Equivalence relation examples
- Related items
  - Equivalence class
  - Partitions

Equivalence relations

- A relation on a set *A* is called an *equivalence relation* if it is reflexive, symmetric, and transitive
  - This is definition 1 in the textbook
- Consider relation *R* = { (*a*,*b*) | *len*(*a*) = *len*(*b*) }
  - Where *len*(*a*) means the length of string *a*
  - It is reflexive: *len*(*a*) = *len*(*a*)
  - It is symmetric: if *len*(*a*) = *len*(b), then *len*(*b*) = *len*(*a*)
  - It is transitive: if *len*(*a*) = *len*(b) and *len*(b) = *len*(c), then *len*(a) = *len*(c)
  - Thus, *R* is a equivalence relation

Equivalence relation example

- Consider the relation $R = \{ (a,b) \mid a \equiv b \pmod{m} \}$
  - Remember that this means that $m \mid a-b$
  - Called "congruence modulo m"
- Is it reflexive: $(a,a) \in R$ means that $m \mid a-a$
  - $a-a = 0$, which is divisible by m
- Is it symmetric: if $(a,b) \in R$ then $(b,a) \in R$
  - (a,b) means that $m \mid a-b$
  - Or that $km = a-b$. Negating that, we get $b-a = -km$
  - Thus, $m \mid b-a$, so $(b,a) \in R$
- Is it transitive: if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
  - (a,b) means that $m \mid a-b$, or that $km = a-b$
  - (b,c) means that $m \mid b-c$, or that $lm = b-c$
  - (a,c) means that $m \mid a-c$, or that $nm = a-c$
  - Adding these two, we get $km+lm = (a-b) + (b-c)$
  - Or $(k+l)m = a-c$
  - Thus, m divides $a-c$, where $n = k+l$
- Thus, congruence modulo m is an equivalence relation

Rosen, section 8.5, question 1

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack
  - { (0,0), (1,1), (2,2), (3,3) }
    - Has all the properties, thus, is an equivalence relation
  - { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
    - Not reflexive: (1,1) is missing
    - Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)
  - { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
    - Has all the properties, thus, is an equivalence relation
  - { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
    - Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)
  - { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
    - Not symmetric: (1,2) is present, but not (2,1)
    - Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

Rosen, section 8.5, question 5

- Suppose that A is a non-empty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x,y) where $f(x) = f(y)$
  - Meaning that x and y are related if and only if $f(x) = f(y)$
- Show that R is an equivalence relation on A

- Reflexivity: $f(x) = f(x)$
  - True, as given the same input, a function always produces the same output
- Symmetry: if $f(x) = f(y)$ then $f(y) = f(x)$
  - True, by the definition of equality
- Transitivity: if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$
  - True, by the definition of equality

Rosen, section 8.5, question 8

- Show that the relation R, consisting of all pairs (x,y) where x and y are bit strings of length three or more that agree except perhaps in their first three bits, is an equivalence relation on the set of all bit strings
- Let $f(x)$ = the bit string formed by the last n-3 bits of the bit string x (where n is the length of the string)
- Thus, we want to show: let R be the relation on A consisting of all ordered pairs (x,y) where $f(x) = f(y)$
- This has been shown in question 5 on the previous slide

Equivalence classes

- Let *R* be an equivalence relation on a set *A*. The set of all elements that are related to an element *a* of *A* is called the *equivalence class* of *a*.
- The equivalence class of *a* with respect to *R* is denoted by $[a]_R$
- When only one relation is under consideration, the subscript is often deleted, and [*a*] is used to denote the equivalence class
- Note that these classes are disjoint!
  - As the equivalence relation is symmetric

More on equivalence classes

- Consider the relation $R = \{(a,b) | a \equiv b \bmod 2\}$
  - Thus, all the even numbers are related to each other
  - As are the odd numbers
- The even numbers form an equivalence class
  - As do the odd numbers
- The equivalence class for the even numbers is denoted by [2] (or [4], or [784], etc.)
  - [2] = { ..., -4, -2, 0, 2, 4, ... }
  - 2 is a *representative* of it's equivalence class
- There are only 2 equivalence classes formed by this equivalence relation

More on equivalence classes

- Consider the relation $R = \{(a,b) \mid a=b \text{ or } a=-b\}$
  - Thus, every number is related to additive inverse
- The equivalence class for an integer *a*:
  - $[7] = \{ 7, -7 \}$
  - $[0] = \{ 0 \}$
  - $[a] = \{ a, -a \}$
- There are an infinite number of equivalence classes formed by this equivalence relation

Partitions

- Consider the relation $R = \{ (a,b) \mid a \equiv b \bmod 2\}$
- This splits the integers into two equivalence classes: even numbers and odd numbers
- Those two sets together form a partition of the integers
- Formally, a partition of a set *S* is a collection of non-empty disjoint subsets of S whose union is *S*
- In this example, the partition is { [0], [1] }
  - Or { {…, -3, -1, 1, 3, …}, {…, -4, -2, 0, 2, 4, …} }

Rosen, section 8.5, question 44

- Which are partitions of the set of integers?
  - The set of even integers and the set of odd integers
    - Yes, it's a valid partition
  - The set of positive integers and the set of negative integers
    - No: 0 is in neither set
  - The set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3
    - Yes, it's a valid partition
  - The set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100
    - Yes, it's a valid partition
  - The set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6
    - The first two sets are not disjoint (2 is in both), so it's not a valid partition