DISCRETE STRUCTURE

LECTURE-2

COMPOSITION OF FUNCTION AND RELATION

TOPICS COVERED

- Function Composition
- Inverse Functions

Function Composition and Inverse Functions

Inverse of addition: u and $-u$

Inverse of multiplication: u and $1/u$

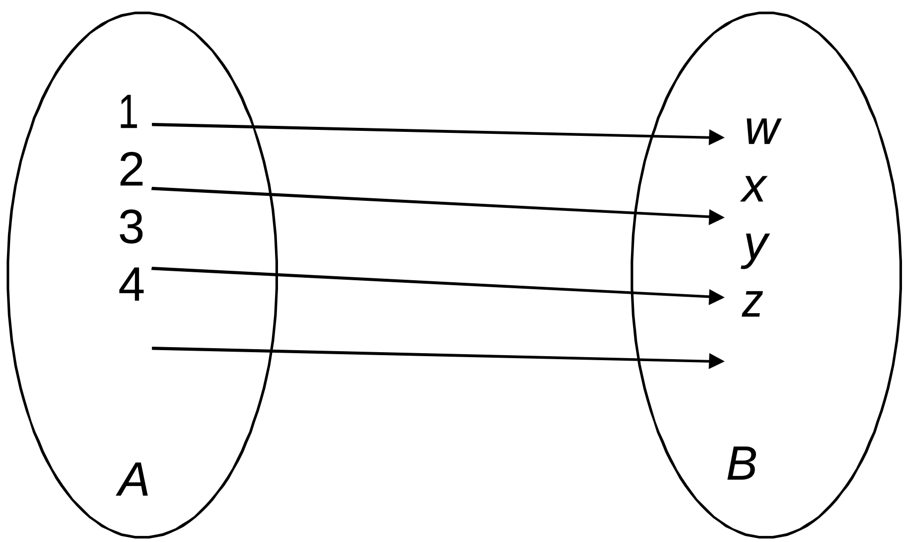
Def : If $f: A \to B$, then f is said to be *bijective*, or to be a *one-to-one correspondence*, if f is both one-to-one and onto.

must be $|A|=|B|$

(if $\infty = \infty$)

but could be

$A \subseteq B$ or $B \subseteq A$



Function Composition and Inverse Functions

Def 1: The function $1_A : A \rightarrow A$, defined by $1_A(a) = a$ for all $a \in A$, is called the identity function for A.

Def. 2: If $f, g : A \rightarrow B$, we say that f and g are equal and write $f = g$, if $f(a) = g(a)$ for all $a \in A$.

Ex. 1 Let $f : \mathrm{Z} \rightarrow \mathrm{Z}$, $g : \mathrm{Z} \rightarrow \mathrm{Q}$ where $f(x) = x = g(x)\ \forall x \in \mathrm{Z}$. Yet, $f \neq g$! f is a one-to-one correspondence, whereas g is one-to-one but not onto.

Ex. 2 $f, g : \mathrm{R} \rightarrow \mathrm{Z}$ defined as follows:

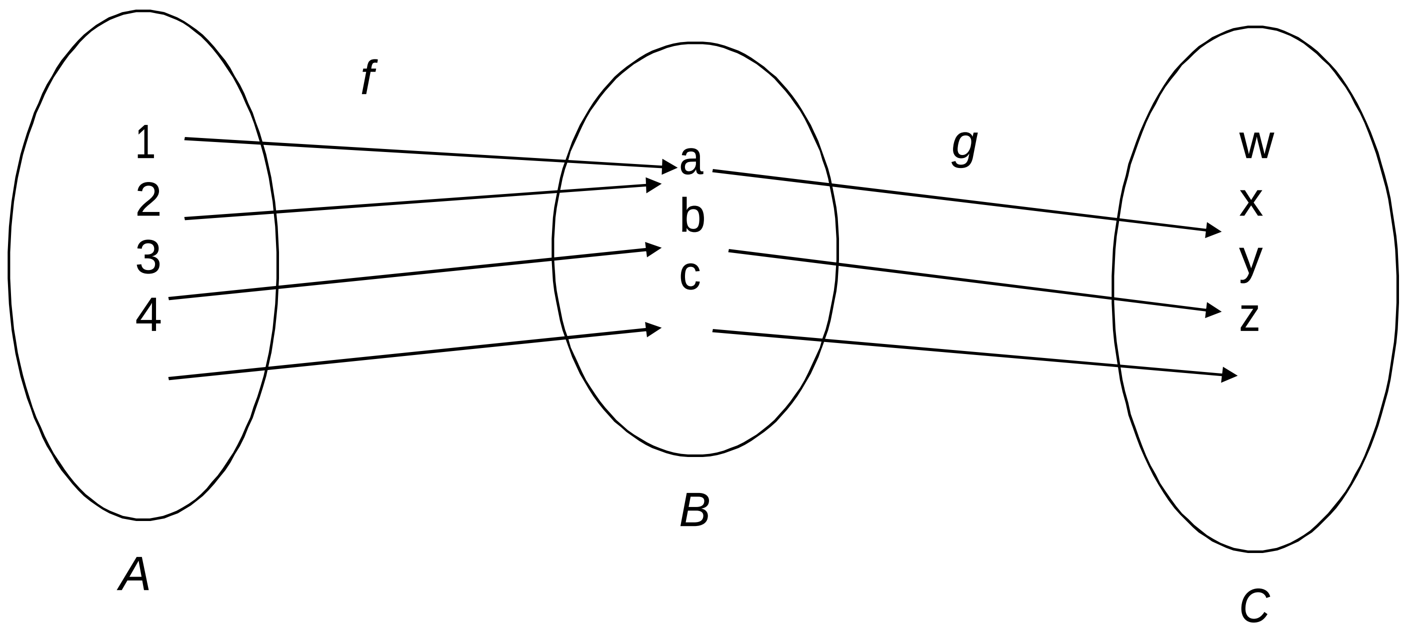
$$f(x) = \begin{cases} x, \text{ if } x \in Z \\ \lfloor x \rfloor + 1, \text{ if } x \in R - Z \end{cases} \quad g(x) = \lceil x \rceil, \text{ for all } x \in \mathrm{R}, \text{ then } f = g.$$

Function Composition and Inverse Functions

Def.: If $f : A \to B$ and $g : B \to C$, we define the composite function, which is denoted $g \circ f : A \to C$, by $(g \circ f)(a) = g(f(a))$, for each $a \in A$.

Ex. 1

$$(g \circ f)(1) = g(f(1)) = g(a) = x,\ gf(2) = x,\ gf(3) = y,\ \text{gf(4) = z}$$



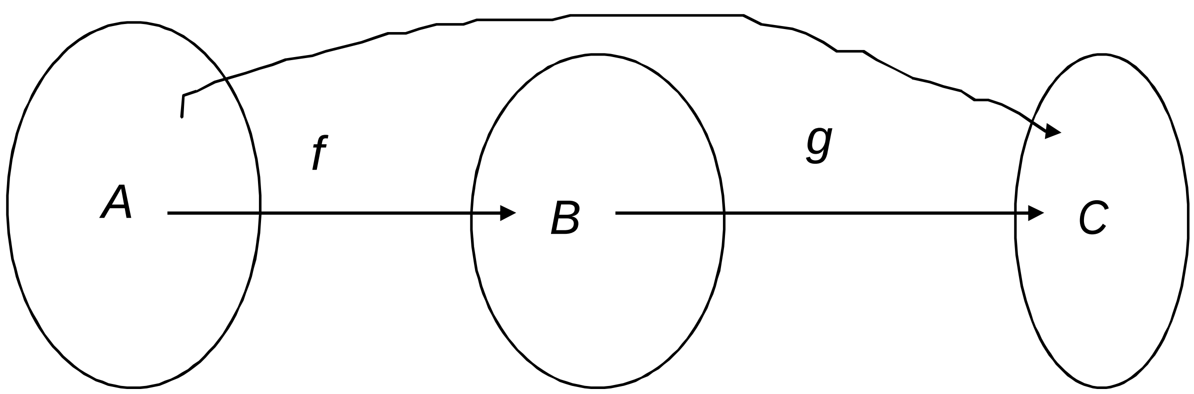
Function Composition and Inverse Functions

Ex. 1: Let $f, g : \mathrm{R} \to \mathrm{R}$ be defined by $f(x) = x^2, g(x) = x + 5$.
Then $g \circ f(x) = g(x^2) = x^2 + 5$, whereas $f \circ g(x) = f(x+5) = (x+5)^2$. Therefore, the composition is not commutative.

Theorem Let $f : A \to B$ and $g : B \to C$

(a) If f, g are one-to-one, then $g \circ f$ is one to one.

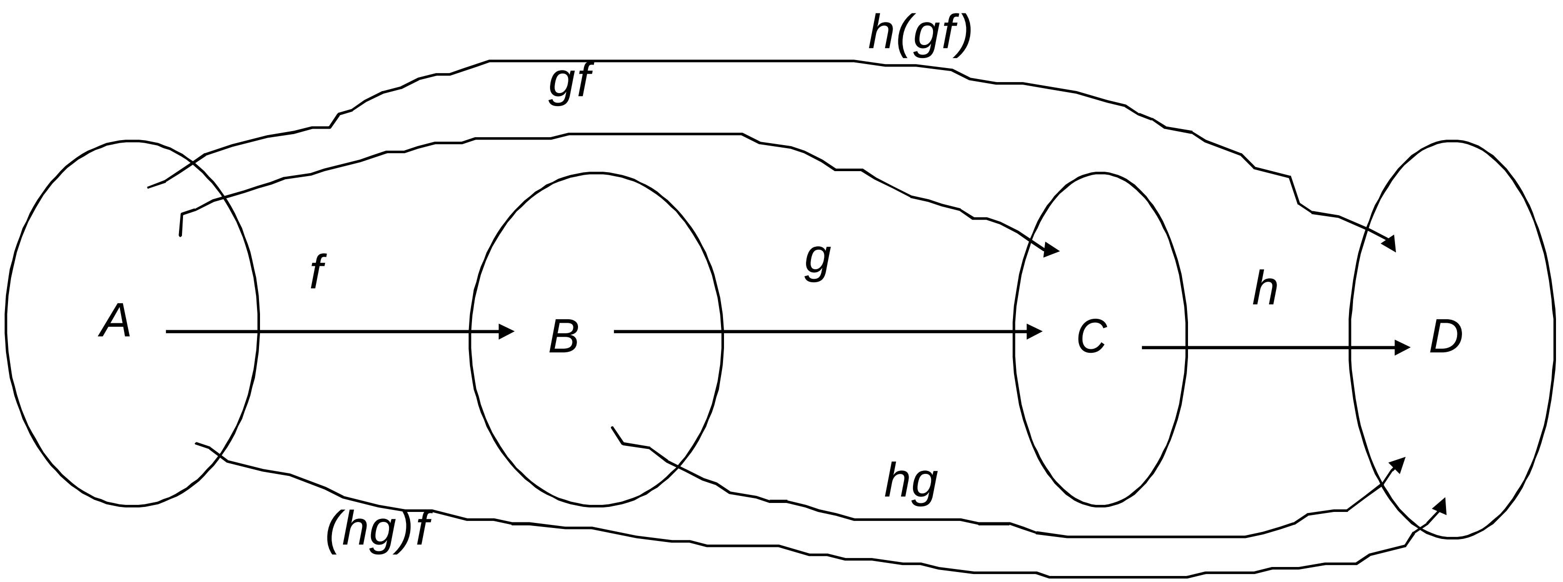
(b) If f, g are onto, then $g \circ f$ is onto.



Function Composition and Inverse Functions

Ex. Let $f, g, h : \mathrm{R} \to \mathrm{R}$ where $f(x) = x^2, g(x) = x + 5, h(x) = \sqrt{x^2 + 2}$. Then $((h \circ g) \circ f)(x) = hg(x^2) = h(x^2 + 5) = \sqrt{(x^2 + 5)^2 + 2} = (h \circ (g \circ f))(x) = h(g(f(x)))$

Theorem If $f : A \to B, g : B \to C, h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

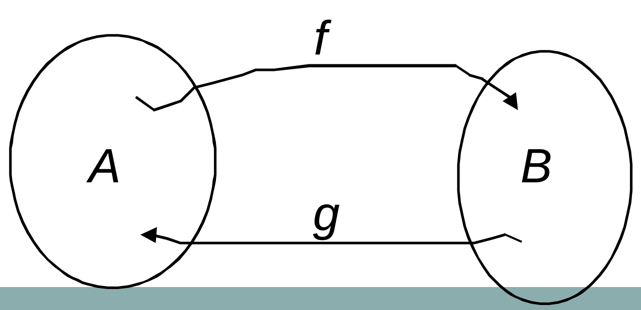


Function Composition and Inverse Functions

Def. If $f : A \to A$, we define $f^{1} = f$, and for $n \in \mathbb{Z}^{+}$, $f^{n+1} = f \circ (f^{n})$.

Ex. $A = \{1,2,3,4\}$, $f : A \to A$ defined by $f = \{(1,2), (2,2), (3,1), (4,3)\}$. Then $f^{2} = f \circ f = \{(1,2), (2,2), (3,2), (4,1)\}$, $f^{3} = \{(1,2), (2,2), (3,2), (4,2)\} = f^{n}, n \geq 3$

Def. If $f : A \to B$, then f is said to be *invertible* if there is a function $g : B \to A$ such that $g \circ f = 1_{A}$ and $f \circ g = 1_{B}$.



Function Composition and Inverse Functions

Ex. $f, g : \mathrm{R} \to \mathrm{R}$, $f(x) = 2x + 5$, $g(x) = \frac{1}{2}(x - 5)$. Then

$gf(x) = g(2x + 5) = \frac{1}{2}(2x + 5 - 5) = x$ and $fg(x) = f(\frac{1}{2}(x - 5))$

$= 2[\frac{1}{2}(x - 5)] + 5 = x$. f and g are both invertible functions

Theorem . invertible function of $f : A \to B$ is unique.

Proof : If $h : B \to A$ is also inverse of f, then $h \circ f = 1_A$ and $f \circ h = 1_B$. Consequently, $h = h \circ 1_B = h \circ (f \circ g) = (h \circ f) \circ g = 1_A \circ g = g$.

(Since it is unique, we use f^{-1} to represent the inverse of f.)

Function Composition and Inverse Functions

Theorem A function $f : A \to B$ is invertible if and only if it is one - to - one and onto.

Theorem If $f : A \to B, g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is invertible and $(\mathrm{g} \circ \mathrm{f})^{-1} = f^{-1} \circ g^{-1}$.

How to find the inverse of a function?

Ex. $f : \mathrm{R} \to \mathrm{R} = \{(x, y) \mid y = mx + b\}$

$f^{-1} = \{(x, y) \mid y = mx + b\}^c = \{(y, x) \mid y = mx + b\} = \{(x, y) \mid x = my + b\}$

$= \{(x, y) \mid y = \dfrac{1}{m}(x - b)\}. \therefore\ f^{-1}(x) = \dfrac{1}{m}(x - b)$

Function Composition and Inverse Functions

Ex. $\mathrm{f} : \mathrm{R} \to \mathrm{R}^{+}, \mathrm{f(x)} = \mathrm{e}^{\mathrm{x}}$. f is *one-to-one and onto.*

$\mathrm{f}^{-1} = \{(x, y) \mid y = e^{x}\}^{c} = \{(y, x) \mid y = e^{x}\} = \{(x, y) \mid x = e^{y}\}$

$= \{(x, y) \mid y = \ln x\}. \therefore\ f^{-1}(x) = \ln x.$

Theorem Let $f : A \to B$ for finite sets A and B, where $|A| = |B|$. Then the following statements are equivalent:

(a) f is one-to-one

(b) f is onto

(c) f is invertible.

Application & Scope of research

Composition of Functions : Word Problems using Composition