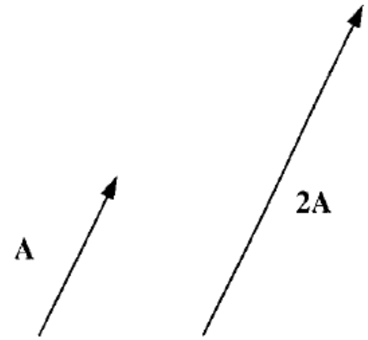
DRONACHARYA College of Engineering

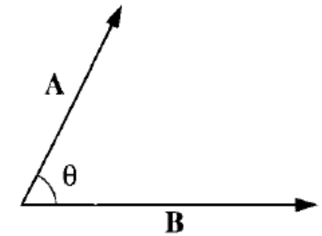
Vector and Tensor

Introduction

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A};$$

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}).$$





$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}.$$

$$\mathbf{A}\cdot\mathbf{B}\equiv AB\cos\theta$$

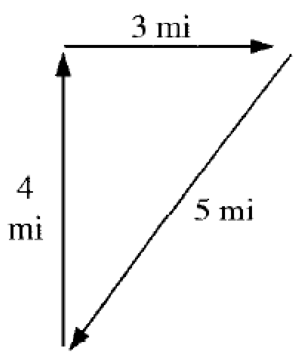
$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$$

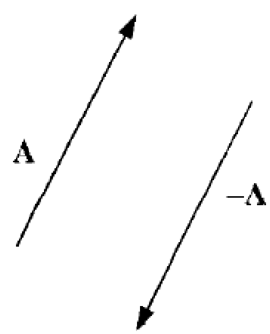
$$\mathbf{A}\times\mathbf{B}\equiv AB\sin\theta\,\hat{\mathbf{n}},$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$$

$$(\mathbf{B}\times\mathbf{A})=-(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times\mathbf{A}=0$$





$$\mathbf{A}+\mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$

$$= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.$$

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}.$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= A_xB_x + A_yB_y + A_zB_z.\end{aligned}$$

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\times(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}.\end{aligned}$$

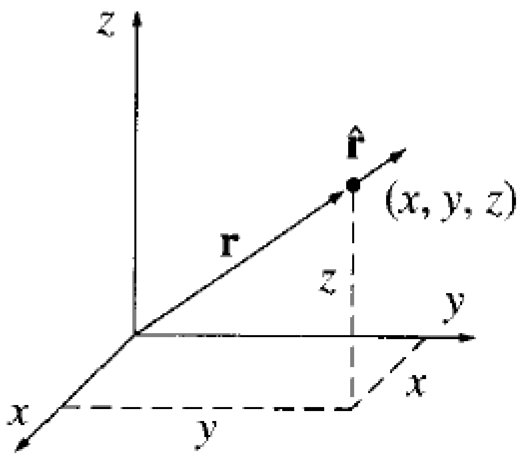
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}. \qquad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

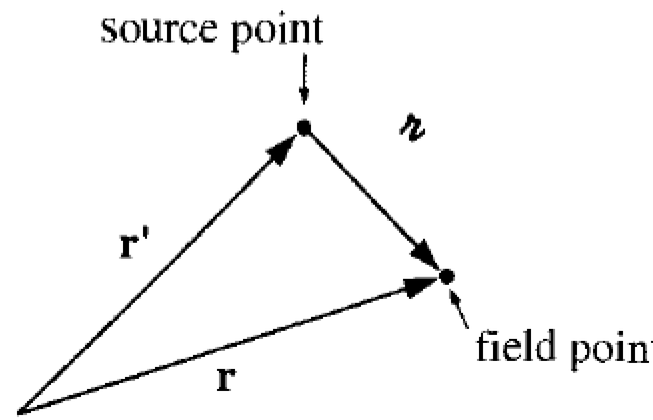
$$\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C});\\ \mathbf{A}\times(\mathbf{B}\times(\mathbf{C}\times\mathbf{D})) &= \mathbf{B}(\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})) - (\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D})\end{aligned}$$

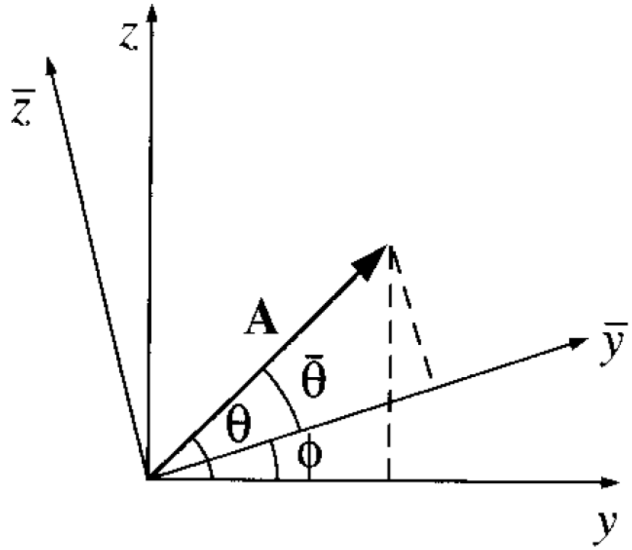




$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}.$$



$$\begin{pmatrix} \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\overline{A}_i = \sum_{j=1}^{3} R_{ij} A_j \qquad \overline{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}.$$

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \qquad \nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Now an ordinary vector **A** can multiply in three ways:

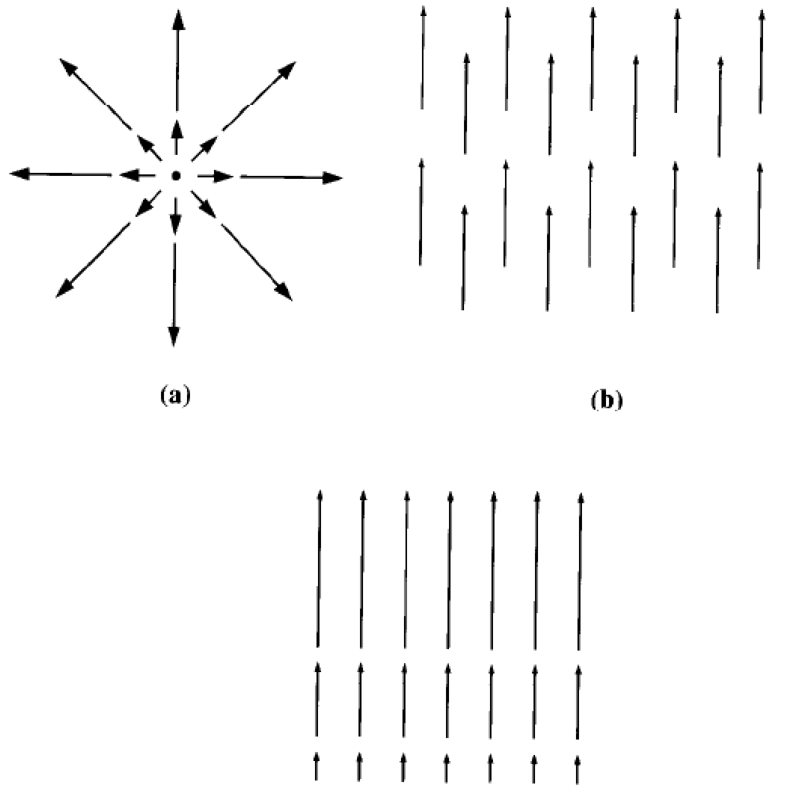
1. Multiply a scalar a : $\mathbf{A}a$;
2. Multiply another vector **B**, via the dot product: $\mathbf{A} \cdot \mathbf{B}$;
3. Multiply another vector via the cross product: $\mathbf{A} \times \mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function **v**, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);
3. On a vector function **v**, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\
&= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.
\end{aligned}$$

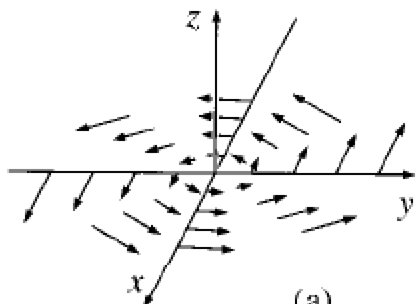
Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question. For example, the vector function in Fig. 1.18a has a large (positive) divergence (if the arrows pointed *in*, it would be a large *negative* divergence), the function in Fig. 1.18b has zero divergence, and the function in Fig. 1.18c again has a positive divergence. (Please understand that $\mathbf{v}$ here is a *function*—there's a different vector associated with every point in space. In the diagrams,



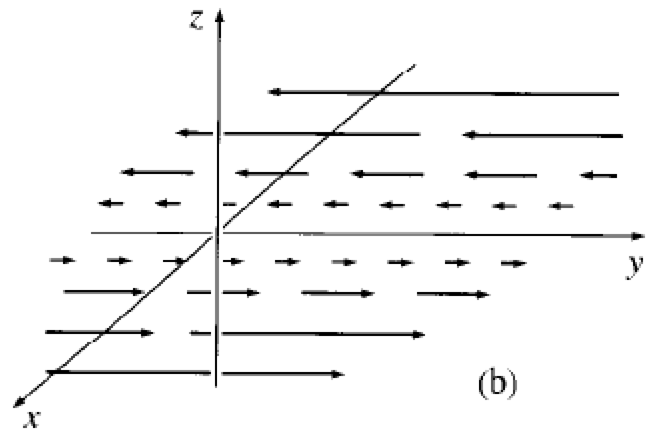
(a) (b) (c)

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



(a)



(b)

Geometrical Interpretation: The name **curl** is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ "curls around" the point in question. Thus the three functions in Fig. 1.18 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 1.19 have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero *curl*. A whirlpool would be a region of large curl.

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$.

(2) Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

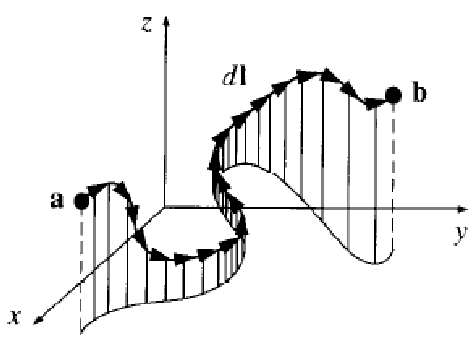
(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$.

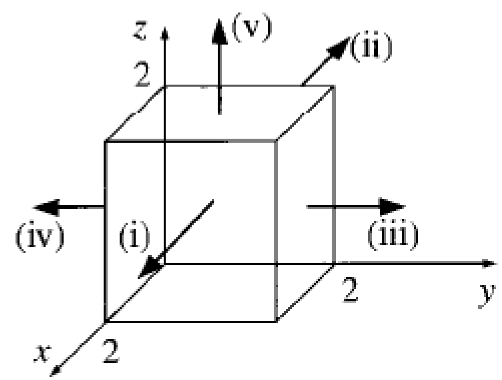
The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

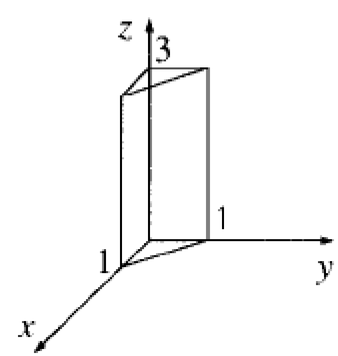
(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

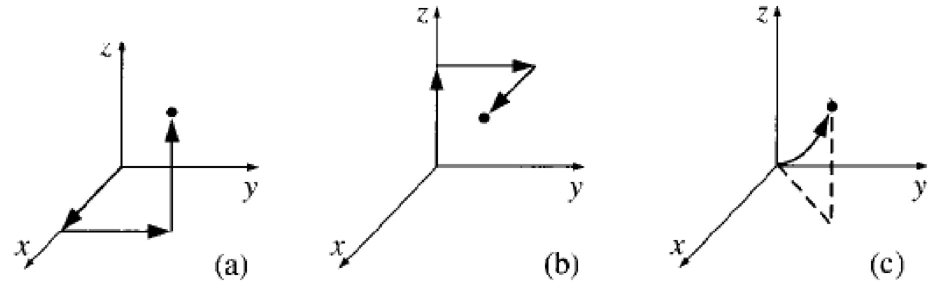
$$\int_{\mathbf{a}\,\mathcal{P}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}, \qquad \int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}, \qquad \int_{\mathcal{V}} T\, d\tau,$$



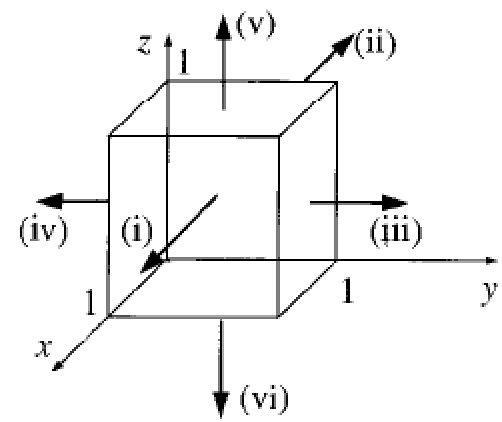




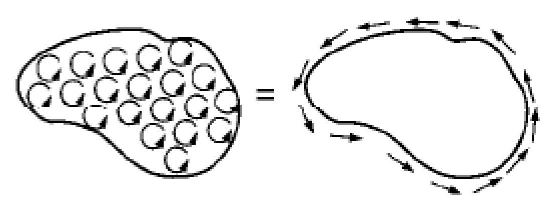
$$\int_{\mathbf{a}\atop\mathcal{P}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).$$



$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a}.$$



$$\int_{\mathcal{S}} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v}\cdot d\mathbf{l}.$$



$$\begin{cases} \delta_{ij} = +1, & \text{if } i = j \\ \delta_{ij} = 0, & \text{if } i \neq j \end{cases}$$

$$\boxed{(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) = \delta_{ij}}$$

$$\begin{cases} \varepsilon_{ijk} = +1, & \text{if } ijk = 123, 231, \text{ or } 312 \\ \varepsilon_{ijk} = -1, & \text{if } ijk = 321, 132, \text{ or } 213 \\ \varepsilon_{ijk} = 0, & \text{if any two indices are alike} \end{cases}$$

$$\boxed{[\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j] = \sum_{k=1}^{3} \varepsilon_{ijk} \boldsymbol{\delta}_k}$$

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{hjk} = 2\delta_{ih}$$

$$\sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$$

$$\begin{aligned} (\mathbf{v} \cdot \mathbf{w}) &= \left(\left\{ \sum_i \boldsymbol{\delta}_i v_i \right\} \cdot \left\{ \sum_j \boldsymbol{\delta}_j w_j \right\} \right) = \sum_i \sum_j (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) v_i w_j \\ &= \sum_i \sum_j \delta_{ij} v_i w_j = \sum_i v_i w_i \end{aligned}$$

$$\begin{aligned} [\mathbf{v} \times \mathbf{w}] &= \left[\left\{ \sum_j \boldsymbol{\delta}_j v_j \right\} \times \left\{ \sum_k \boldsymbol{\delta}_k w_k \right\} \right] \\ &= \sum_j \sum_k [\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k] v_j w_k = \sum_i \sum_j \sum_k \varepsilon_{ijk} \boldsymbol{\delta}_i v_j w_k \\ &= \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

$$(\mathbf{u} \cdot [\mathbf{v} \times \mathbf{w}]) = \sum_i u_i [\mathbf{v} \times \mathbf{w}]_i = \sum_i \sum_j \sum_k \varepsilon_{ijk} u_i v_j w_k$$

$$(\mathbf{u} \cdot [\mathbf{v} \times \mathbf{w}]) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$[\mathbf{u} \times [\mathbf{v} \times \mathbf{w}]]_i = \sum_j \sum_k \varepsilon_{ijk} u_j [\mathbf{v} \times \mathbf{w}]_k = \sum_j \sum_k \varepsilon_{ijk} u_j \left\{ \sum_l \sum_m \varepsilon_{klm} v_l w_m \right\}$$

$$= \sum_j \sum_k \sum_l \sum_m \varepsilon_{ijk} \varepsilon_{klm} u_j v_l w_m = \sum_j \sum_k \sum_l \sum_m \varepsilon_{ijk} \varepsilon_{lmk} u_j v_l w_m$$

$$(\boldsymbol{\delta}_i \boldsymbol{\delta}_j : \boldsymbol{\delta}_k \boldsymbol{\delta}_l) = (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) = \delta_{jk} \delta_{il}$$

$$[\boldsymbol{\delta}_i \boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k] = \boldsymbol{\delta}_i (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) = \boldsymbol{\delta}_i \delta_{jk}$$

$$[\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j \boldsymbol{\delta}_k] = (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \boldsymbol{\delta}_k = \delta_{ij} \boldsymbol{\delta}_k$$

$$\{\boldsymbol{\delta}_i \boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k \boldsymbol{\delta}_l\} = \boldsymbol{\delta}_i (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) \boldsymbol{\delta}_l = \delta_{jk} \boldsymbol{\delta}_i \boldsymbol{\delta}_l$$

$$\{\boldsymbol{\delta}_i \boldsymbol{\delta}_j \times \boldsymbol{\delta}_k\} = \boldsymbol{\delta}_i [\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k] = \sum_{l=1}^{3} \varepsilon_{jkl} \boldsymbol{\delta}_i \boldsymbol{\delta}_l$$

$$\{\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j \boldsymbol{\delta}_k\} = [\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j] \boldsymbol{\delta}_k = \sum_{l=1}^{3} \varepsilon_{ijk} \boldsymbol{\delta}_l \boldsymbol{\delta}_k$$

$$\begin{aligned} \boldsymbol{\tau} = {} & \boldsymbol{\delta}_1 \boldsymbol{\delta}_1 \tau_{11} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_2 \tau_{12} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_3 \tau_{13} \\ & + \boldsymbol{\delta}_2 \boldsymbol{\delta}_1 \tau_{21} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_2 \tau_{22} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_3 \tau_{23} \\ & + \boldsymbol{\delta}_3 \boldsymbol{\delta}_1 \tau_{31} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_2 \tau_{32} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_3 \tau_{33} \\ = {} & \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij} \end{aligned}$$

$$\boldsymbol{\sigma} + \boldsymbol{\tau} = \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \sigma_{ij} + \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij} = \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j (\sigma_{ij} + \tau_{ij})$$

$$s\boldsymbol{\tau} = s\left\{\sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij}\right\} = \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \{s\tau_{ij}\}$$

$$(\boldsymbol{\sigma}:\boldsymbol{\tau}) = \left(\left\{\sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \sigma_{ij}\right\}:\left\{\sum_k \sum_l \boldsymbol{\delta}_k \boldsymbol{\delta}_l \tau_{kl}\right\}\right) = \sum_i \sum_j \sum_k \sum_l (\boldsymbol{\delta}_i \boldsymbol{\delta}_j : \boldsymbol{\delta}_k \boldsymbol{\delta}_l) \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_j \sum_k \sum_l \delta_{il} \delta_{jk} \sigma_{ij} \tau_{kl} = \sum_i \sum_j \sigma_{ij} \tau_{ji}$$

$$(\boldsymbol{\tau}:\mathbf{vw}) = \sum_i \sum_j \tau_{ij} v_j w_i$$

$$(\mathbf{uv}:\mathbf{wz}) = \sum_i \sum_j u_i v_j w_j z_i$$

$$\{\boldsymbol{\sigma} \cdot \boldsymbol{\tau}\} = \left\{\left(\sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \sigma_{ij}\right) \cdot \left(\sum_k \sum_l \boldsymbol{\delta}_k \boldsymbol{\delta}_l \tau_{kl}\right)\right\} = \sum_i \sum_j \sum_k \sum_l \{\boldsymbol{\delta}_i \boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k \boldsymbol{\delta}_l\} \sigma_{ij} \tau_{kl}$$

$$= \sum_i \sum_j \sum_k \sum_l \delta_{jk} \boldsymbol{\delta}_i \boldsymbol{\delta}_l \sigma_{ij} \tau_{kl} = \sum_i \sum_l \boldsymbol{\delta}_i \boldsymbol{\delta}_l \left(\sum_j \sigma_{ij} \tau_{jl}\right)$$

$$[\boldsymbol{\tau} \cdot \mathbf{v}] = \left[\left\{\sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij}\right\} \cdot \left\{\sum_k \boldsymbol{\delta}_k v_k\right\}\right] = \sum_i \sum_j \sum_k [\boldsymbol{\delta}_i \boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k] \tau_{ij} v_k$$

$$= \sum_i \sum_j \sum_k \boldsymbol{\delta}_i \delta_{jk} \tau_{ij} v_k = \sum_i \boldsymbol{\delta}_i \left\{\sum_j \tau_{ij} v_j\right\}$$

$$\{\boldsymbol{\tau} \times \mathbf{v}\} = \left\{\left(\sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij}\right) \times \left(\sum_k \boldsymbol{\delta}_k v_k\right)\right\} = \sum_i \sum_j \sum_k [\boldsymbol{\delta}_i \boldsymbol{\delta}_j \times \boldsymbol{\delta}_k] \tau_{ij} v_k$$

$$= \sum_i \sum_j \sum_k \sum_l \varepsilon_{jkl} \boldsymbol{\delta}_i \boldsymbol{\delta}_l \tau_{ij} v_k = \sum_i \sum_l \boldsymbol{\delta}_i \boldsymbol{\delta}_l \left\{\sum_j \sum_k \varepsilon_{jkl} \tau_{ij} v_k\right\}$$

$$(\nabla \cdot \mathbf{v}) = \left(\left\{\sum_i \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right\} \cdot \left\{\sum_j \boldsymbol{\delta}_j v_j\right\}\right) = \sum_i \sum_j (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} v_j$$

$$= \sum_i \sum_j \delta_{ij} \frac{\partial}{\partial x_i} v_j = \sum_i \frac{\partial v_i}{\partial x_i}$$

$$\nabla \mathbf{v} = \left\{\sum_i \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right\}\left\{\sum_j \boldsymbol{\delta}_j v_j\right\} = \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{\partial}{\partial x_i} v_j$$

$$[\nabla \times \mathbf{v}] = \left[\left\{\sum_j \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right\} \times \left\{\sum_k \boldsymbol{\delta}_k v_k\right\}\right]$$

$$= \sum_j \sum_k [\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k] \frac{\partial}{\partial x_j} v_k = \sum_i \sum_j \sum_k \varepsilon_{ijk} \boldsymbol{\delta}_i \frac{\partial}{\partial x_j} v_k$$

$$= \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \boldsymbol{\delta}_1\left\{\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right\} + \boldsymbol{\delta}_2\left\{\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right\} + \boldsymbol{\delta}_3\left\{\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right\}$$

$$(\nabla \mathbf{v})^{\dagger} = \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} v_i$$

$$[\nabla \cdot \boldsymbol{\tau}] = \left[\left\{\sum_i \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right\} \cdot \left\{\sum_j \sum_k \boldsymbol{\delta}_j \boldsymbol{\delta}_k \tau_{jk}\right\}\right] = \sum_i \sum_j \sum_k [\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j \boldsymbol{\delta}_k] \frac{\partial}{\partial x_i} \tau_{jk}$$

$$= \sum_i \sum_j \sum_k \delta_{ij} \boldsymbol{\delta}_k \frac{\partial}{\partial x_i} \tau_{jk} = \sum_k \boldsymbol{\delta}_k \left\{\sum_i \frac{\partial}{\partial x_i} \tau_{ik}\right\}$$

$$(\nabla \cdot \nabla s) = \left(\left\{\sum_i \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right\} \cdot \left\{\sum_j \boldsymbol{\delta}_j \frac{\partial s}{\partial x_j}\right\}\right)$$

$$= \sum_i \sum_j \delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial s}{\partial x_j} = \left\{\sum_i \frac{\partial^2}{\partial x_i^2} s\right\}$$

$$
\begin{aligned}
[\nabla \cdot \nabla \mathbf{v}] &= \left[\left\{\sum_i \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right\} \cdot \left\{\sum_j \sum_k \boldsymbol{\delta}_j \boldsymbol{\delta}_k \frac{\partial}{\partial x_j} v_k\right\}\right] \\
&= \sum_i \sum_j \sum_k [\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j \boldsymbol{\delta}_k] \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v_k \\
&= \sum_i \sum_j \sum_k \delta_{ij} \boldsymbol{\delta}_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v_k = \sum_k \boldsymbol{\delta}_k \left(\sum_i \frac{\partial^2}{\partial x_i^2} v_k\right)
\end{aligned}
$$

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{A}$$

$$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \tag{B}$$

$$\begin{aligned}(\boldsymbol{\tau}:\nabla \mathbf{v}) = {} & \tau_{xx}\left(\frac{\partial v_x}{\partial x}\right) + \tau_{xy}\left(\frac{\partial v_x}{\partial y}\right) + \tau_{xz}\left(\frac{\partial v_x}{\partial z}\right) \\ & + \tau_{yx}\left(\frac{\partial v_y}{\partial x}\right) + \tau_{yy}\left(\frac{\partial v_y}{\partial y}\right) + \tau_{yz}\left(\frac{\partial v_y}{\partial z}\right) \\ & + \tau_{zx}\left(\frac{\partial v_z}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v_z}{\partial y}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right)\end{aligned} \tag{C}$$

$$[\nabla s]_x = \frac{\partial s}{\partial x} \tag{D}$$

$$[\nabla s]_y = \frac{\partial s}{\partial y} \tag{E}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \tag{F}$$

$$[\nabla \times \mathbf{v}]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \tag{G}$$

$$[\nabla \times \mathbf{v}]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \tag{H}$$

$$[\nabla \times \mathbf{v}]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{I}$$

$$[\nabla \cdot \boldsymbol{\tau}]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{J}$$

$$[\nabla \cdot \boldsymbol{\tau}]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{K}$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{L}$$

$$[\nabla^2 \mathbf{v}]_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \tag{M}$$

$$[\nabla^2 \mathbf{v}]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \tag{N}$$

$$[\nabla^2 \mathbf{v}]_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \tag{O}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_x = v_x\left(\frac{\partial w_x}{\partial x}\right) + v_y\left(\frac{\partial w_x}{\partial y}\right) + v_z\left(\frac{\partial w_x}{\partial z}\right) \tag{P}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_y = v_x\left(\frac{\partial w_y}{\partial x}\right) + v_y\left(\frac{\partial w_y}{\partial y}\right) + v_z\left(\frac{\partial w_y}{\partial z}\right) \tag{Q}$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_z = v_x\left(\frac{\partial w_z}{\partial x}\right) + v_y\left(\frac{\partial w_z}{\partial y}\right) + v_z\left(\frac{\partial w_z}{\partial z}\right) \tag{R}$$

$$\{\nabla \mathbf{v}\}_{xx} = \frac{\partial v_x}{\partial x} \quad \text{(S)}$$

$$\{\nabla \mathbf{v}\}_{xy} = \frac{\partial v_y}{\partial x} \quad \text{(T)}$$

$$\{\nabla \mathbf{v}\}_{xz} = \frac{\partial v_z}{\partial x} \quad \text{(U)}$$

$$\{\nabla \mathbf{v}\}_{yx} = \frac{\partial v_x}{\partial y} \quad \text{(V)}$$

$$\{\nabla \mathbf{v}\}_{yy} = \frac{\partial v_y}{\partial y} \quad \text{(W)}$$

$$\{\nabla \mathbf{v}\}_{yz} = \frac{\partial v_z}{\partial y} \quad \text{(X)}$$

$$\{\nabla \mathbf{v}\}_{zx} = \frac{\partial v_x}{\partial z} \quad \text{(Y)}$$

$$\{\nabla \mathbf{v}\}_{zy} = \frac{\partial v_y}{\partial z} \quad \text{(Z)}$$

$$\{\nabla \mathbf{v}\}_{zz} = \frac{\partial v_z}{\partial z} \quad \text{(AA)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xx} = (\mathbf{v} \cdot \nabla)\tau_{xx} \quad \text{(BB)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xy} = (\mathbf{v} \cdot \nabla)\tau_{xy} \quad \text{(CC)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{xz} = (\mathbf{v} \cdot \nabla)\tau_{xz} \quad \text{(DD)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yx} = (\mathbf{v} \cdot \nabla)\tau_{yx} \quad \text{(EE)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yy} = (\mathbf{v} \cdot \nabla)\tau_{yy} \quad \text{(FF)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{yz} = (\mathbf{v} \cdot \nabla)\tau_{yz} \quad \text{(GG)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zx} = (\mathbf{v} \cdot \nabla)\tau_{zx} \quad \text{(HH)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zy} = (\mathbf{v} \cdot \nabla)\tau_{zy} \quad \text{(II)}$$

$$\{\mathbf{v} \cdot \nabla \boldsymbol{\tau}\}_{zz} = (\mathbf{v} \cdot \nabla)\tau_{zz} \quad \text{(JJ)}$$

Navier-Stokes Equation

- Navier-Stokes equations for incompressible flow involve four basic quantities
  - Local (unsteady) acceleration
  - Convective acceleration
  - Pressure gradients
  - Viscous forces

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u}\cdot\nabla \boldsymbol{u} = -\nabla p + \mu\nabla^2 \boldsymbol{u}$$

Flow	Local Acceleration	Convective Acceleration	Pressure Gradients	Viscous Forces
Pipe (steady laminar flow)			✓	✓
Plate (impulsively started)	✓			✓
Plate (boundary layer)		✓		✓
Airfoil (inviscid)		✓	✓	
Pipe (inviscid, impulsively started)	✓		✓	
Cylinder (steady viscous flow)		✓	✓	✓
Airfoil (unsteady flow)	✓	✓	✓	✓

✓ *Means term is important for this flow.*

Classical Mechanics (Newton)	**Quantum Mechanics** (Bohr, Heisenberg, Schrödinger, *et al.*)
Special Relativity (Einstein)	**Quantum Field Theory** (Dirac, Pauli, Feynman, Schwinger, *et al.*)

Limits

- Newtonian Mechanics
  - Very good for everyday life
  - Fails at high speeds ~ speed of light
- So it must be replaced by Special theory of Relativity
  - Introduced by Einstein in 1905
  - Applicable for smaller objects with higher speeds
- It is superseded by Quantum mechanics because of certain limitations
- Relativistic Quantum mechanics is known as
  - Quantum field theory

History

- **Archimedes (285–212 B.C.)** formulated the principles of buoyancy of submerged bodies and determined the gold content of the crown of King Hiero I
- At about the same time, the **Roman engineers** built an extensive network of fresh-water supply
- The development of fluid mechanics continued along two different paths:
  - **mathematicians and physicists developed the theory** and applied it to “idealized” problems that did not have much practical value
  - **engineers developed empirical equations** that could be used in the design of fluid systems in a limited range.
- The **lack of communication** between these two groups hindered the development of fluid mechanics for a long time.

Modern Times

- **Leonardo da Vinci (1459–1519)** conducted several experiments and derived the conservation of mass equation for one-dimensional steady flow
- The development of the **laws of motion by Isaac Newton (1649–1727)** and the linear law of viscosity for the so-called Newtonian fluids set the stage for advances in fluid mechanics.
- **Leonhard Euler (1707–1783) obtained the differential equations for fluid motion** in 1755.
- **Daniel Bernoulli (1700–1782) developed the energy equation** for incompressible flow in 1738.
- **Lord Rayleigh (1849–1919) developed the powerful dimensional analysis technique**.
- **Osborn Reynolds (1849–1912) conducted extensive experiments with pipe flow** and in 1883 came up with the dimensionless number that bears his name.
- The general equations of fluid motion that include the effects of fluid friction, known as the Navier–Stokes equations, were developed by **Claude Louis Marie Navier (1785–1836) in 1827 and independently by George Gabriel Stokes** (1819–1903) in 1845.
- **Ludwig Prandtl (1875–1953) showed that fluid flows can be divided into a layer near the walls, called the boundary layer, where the friction effects** are significant and an outer layer where such effects are negligible, thus the Euler and Bernoulli equations are applicable.
- **Theodore von Karman (1889–1963) and Sir Geoffrey I. Taylor (1886–1975)** also contributed greatly to the development of fluid mechanics in the twentieth century.
- The **availability of high-speed computers in the last decades and the development of numerical methods** have made it possible to solve a variety of real-world fluids problems and to conduct design and optimization studies through numerical simulation.