Lecture-11

DIFFRACTION

Unit-01

DIFFRACTION

FRAUNHOFER DIFFRACTION BY A SINGLE SLIT

Conditions of Maxima and Minima

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FRAUNHOFER DIFFRACTION BY A SINGLE SLIT

- Since in Fraunhofer diffraction the source is effectively at infinite distance, a collimated parallel beam of monochromatic light of wavelength λ can be taken as incident normally on a narrow slit AB of width b.
- We divide this wavefront into a large number of points n each sending waves of equal amplitude a according to Huygens' principle. These waves get diffracted and then interfere to produce diffraction pattern on the screen.
- The secondary waves that travel along the direction of incident beam are focused at point C while those inclined at an angle θ with the direction of incident beam (due to diffraction) are focused at another point P .
- To find out the resultant intensity at P, we draw a perpendicular AK on BK. It is clear from the figure that the optical paths of the waves traveled after the plane AK to the point P are equal. However, the optical paths of the waves originating from points on AB (from A toward B) and reaching the point P gradually increase.

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- Hence, the phase difference between them gets larger. This is shown in Fig. for n number of waves each of amplitude a.
- **In this figure a phase difference of φ is taken between two successive waves. The resultant amplitude of these waves at point P is shown as R.**
- The total path difference between the waves originating from extreme points A and B is BK = AB sinθ = b sinθ.
- Therefore, the path difference between different waves originating from all the points of the slit AB vary between zero and b sinθ.
- The phase difference corresponding to path difference b sinθ will be (2π/λ) b sin θ

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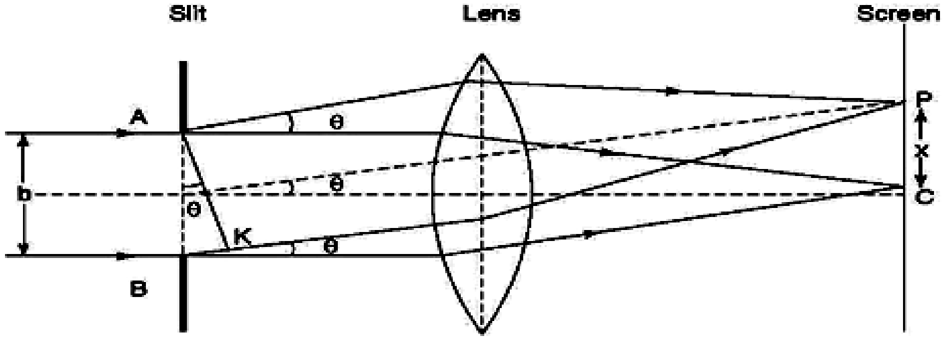
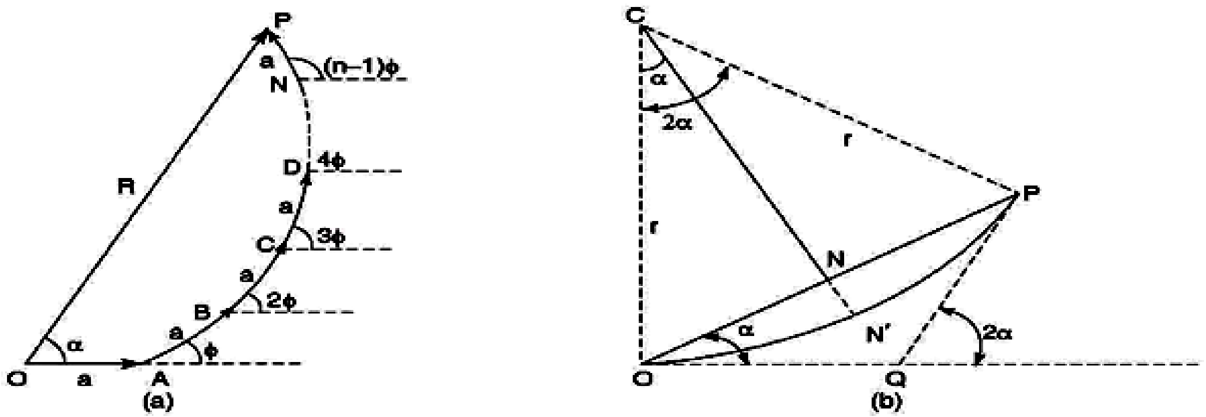


Since the aperture is divided into n equal parts, the phase difference between any two consecutive parts will be $\frac{1}{n}\frac{2\pi}{\lambda} b \sin\theta \; (= \phi, \text{say})$.



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- The resultant amplitude and intensity at point P due to all these secondary waves can be obtained by vector polygon method.
- Let α be the phase difference between the waves from the initial direction to the resultant, then 2α will be the total phase difference between the secondary waves originating from extreme points of the slit AB.
- Here, it is taken that all the amplitudes constitute an arc due to their large number and small phase difference between them. Because of the symmetry, we have ∠ O = α and ∠Q = 2α.
- The chord OP gives the resultant amplitude due to all the secondary waves at point P.

$$\sin \alpha = \frac{ON}{OC} = \frac{ON}{r}$$

$$\text{or } ON = r \sin \alpha \qquad \text{(i)}$$

where, r is the radius of the circular arc.

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$\therefore$ Chord OP = 2ON = 2 $r \sin\alpha$

$\therefore$ Chord OP = resultant amplitude

$\therefore R = 2r\sin\alpha$

The length of the arc ON′P = na, where n is an integer number and a is the amplitude of each vibration (Fig. 2.9 b).

We know that,

$$\angle PCO = 2\alpha = \frac{\text{Arc ON'P}}{\text{Radius}} = \frac{na}{r}$$

$$\text{or } 2r = \frac{na}{\alpha} \qquad \text{(iii)}$$

Substituting the value of $2r$ in Eq. (ii), we get

$$R = na\frac{\sin\alpha}{\alpha}$$

$$R = A_0\frac{\sin\alpha}{\alpha} \qquad \text{(iv)}$$

Here we have taken $na = A_0$. Thus, resultant intensity at point P on the screen is given by a measure of

$$I = R^2 = A_0^2\frac{\sin^2\alpha}{\alpha^2} = I_0\left[\frac{\sin\alpha}{\alpha}\right]^2$$

$$I = I_0\left[\frac{\sin\alpha}{\alpha}\right]^2 \qquad \text{(v)}$$

Thus, the magnitude of the resultant intensity at any point on the screen is a function of α and the slit width b. Since the phase difference of 2α is introduced due to the path difference of $b\sin\theta$.

$$\therefore \quad 2\alpha = \frac{2\pi}{\lambda}\times b\sin\theta$$

$$\text{or } \alpha = \frac{\pi}{\lambda}\times b\sin\theta$$

Conditions of Maxima and Minima

It is clear from Eq. (iv) that the resultant amplitude R will be a maximum when

$$\frac{\sin\alpha}{\alpha} = 1, \text{ which is true when } \alpha \to 0, \text{ i.e.,}$$

$$\frac{\pi}{\lambda} b \sin\theta \to 0 \text{ or } \sin\theta \to 0$$

$$\Rightarrow \theta = 0^\circ$$

From Eq. (iv),

$$R = \frac{A_0}{\alpha}\sin\alpha = \frac{A_0}{\alpha}\left[\alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \ldots\right]$$

$$= A_0\left[1 - \frac{\alpha^2}{\lfloor 3} + \frac{\alpha^4}{\lfloor 5} + \frac{\alpha^6}{\lfloor 7} + \ldots\right]$$

$$R = A_0 \text{ for } \alpha = 0.$$

Hence the intensity corresponding to $\alpha = 0$ is $I = R^2 = A_0^2 = I_0$. This is called as the intensity of central (principal) maximum.

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The first relation $y = \alpha$ represents the equation of straight line passing through the origin making an angle 45° with the axis and the equation $y = \tan\alpha$ represents a discontinuous curve having a number of branches with asymptotes at the intervals of π (Fig. 2.10). The points of intersections of these curves will give the values of α that will satisfy the relation $\alpha = \tan\alpha$.

Therefore, the maxima occur when

$$\alpha = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots \text{ or } \alpha = (2n+1)\frac{\pi}{2}, \quad n = 1, 2, 3 \ldots$$

These are called points of secondary maxima. A measure of intensity of first secondary maxima is obtained from Eq. (v) with $\alpha = \frac{3\pi}{2}$, as

$$I_1 = A_0^2 \left[\frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}}\right]^2 = \frac{4}{9\pi^2} I_0$$

Similarly, the intensity of second secondary maxima is

$$I_2 = A^2 \left[\frac{\sin\left(\frac{5\pi}{2}\right)}{\frac{5\pi}{2}}\right]^2 = \frac{4}{25\pi^2} I_0$$

Similarly, $$I_3 = A^2 \left[\frac{\sin\left(\frac{7\pi}{2}\right)}{\frac{7\pi}{2}}\right]^2 = \frac{4}{49\pi^2} I_0$$ and so on.

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In order to determine the position of maximum intensity, let us differentiate Eq (v) w.r.t. to α and equate it to zero, i.e.

$$\frac{dI}{d\alpha} = 2I_0 \frac{\sin\alpha}{\alpha}\left[\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right] = 0$$

Since I_0 cannot be zero, either $\sin\alpha = 0$ or $\alpha\cos\alpha - \sin\alpha = 0$. The equation $\sin\alpha = 0$ determines the positions of minima [Eq. (iv)] except when $\alpha = 0$ because it corresponds to the position of maximum. Therefore, the following condition should be satisfied for obtaining the minima

$$\sin\alpha = 0 \quad \text{or} = \alpha = \pm m\pi$$

or

$$\frac{\pi}{\lambda} b \sin\theta = \pm m\pi$$

or

$$b\sin\theta = \pm m\lambda,\ m = 1, 2, 3\ldots$$

The position of maxima are given by

$$\alpha\cos\alpha - \sin\alpha = 0$$

$$\text{or } \alpha = \tan\alpha$$

This equation can be solved graphically by plotting the curves

$$y = \alpha$$

$$\text{and } y = \tan\alpha$$

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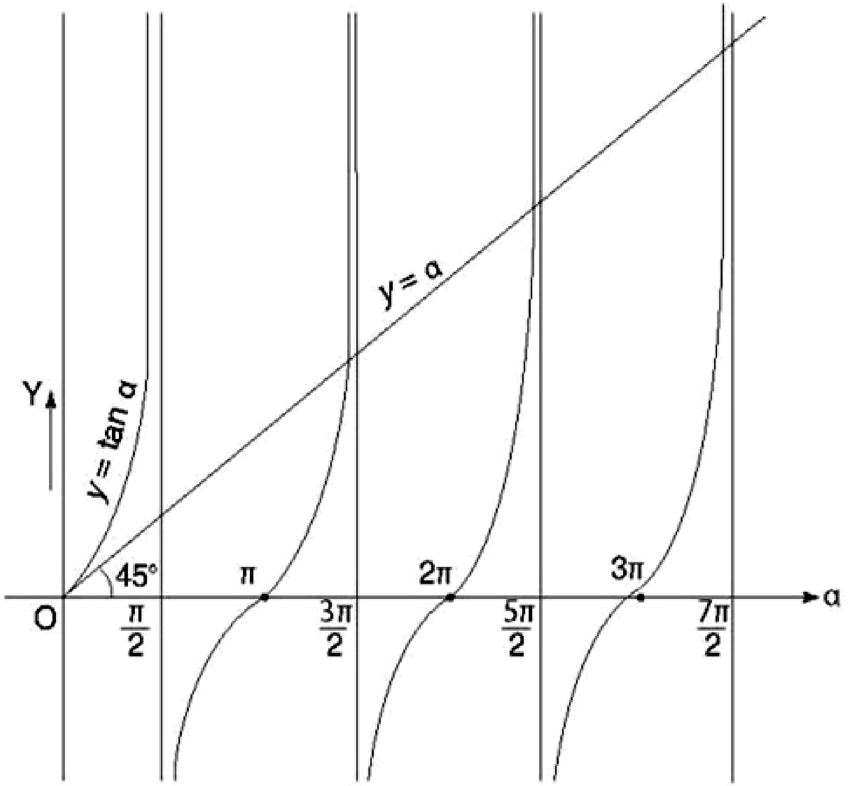
Thus, the ratio of relative intensities of successive maxima are

$$1:\frac{4}{9\pi^2}:\frac{4}{25\pi^2}:\frac{4}{49\pi^2}:\ldots$$

The intensity of the first secondary maxima $\frac{4}{9\pi^2}$, i.e., 4.5% that of principal maximum as shown in Fig. 2.11.



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