Section -D

Single and multiple integral

INTEGRAL CALCULUS

- Applications of single integration to find volume of solids and surface area of solids of revolution.
- Double integral, change of order of integration
- Double integral in polar coordinates,
- Applications of double integral to find area enclosed by plane curves,
- Triple integral
- Change of variables, volume of solids,
- Dirichlet's integral.
- Beta and gamma functions and relationship between them

Applications of Single Integration

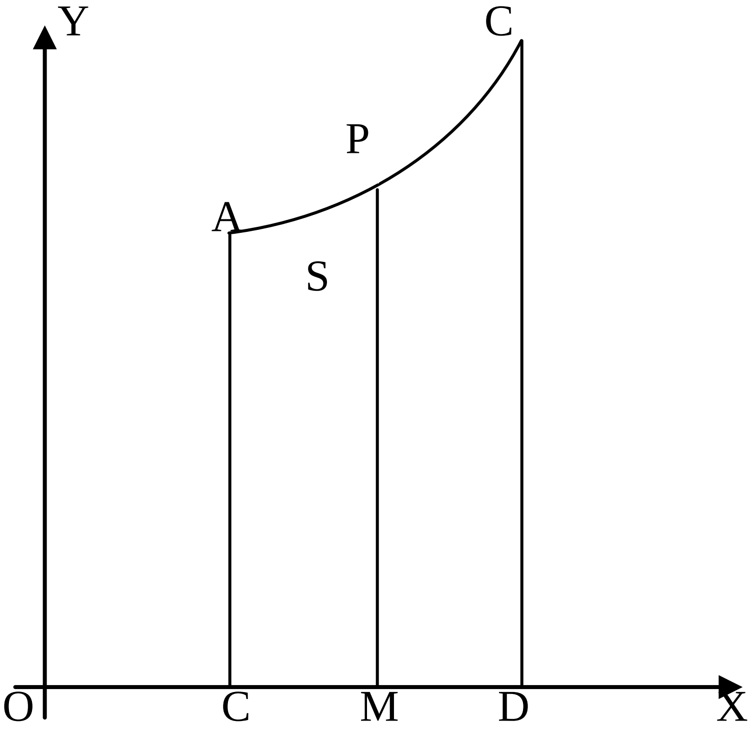
Volume Formulae for Cartesian Equations

Revolution About x-axis:- The Volume of a solid generated by the revolution about the x-axis of the area bounded by the curve y=f(x), the x-axis and the ordinates x=a, x=b is $\int_a^b \pi y^2 dx$

Revolution About y-axis:- The volume of the solid generated by the revolution about the y-axis of the area bounded by the curve x=f(y), the x-axis and the ordinates y=a, y=b is $\int_a^b \pi x^2 dy$

Revolution About Any axis:- The volume of the solid generated by the revolution about any axis CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC,BD on the axis, is

$\int_{OC}^{OD} \pi(PM)^2 d\,(OM)$ where O is a fixed point on the axis CD and PM is Perpendicular from any point P of the curve AB on CD.



Volume Formulae for Parametric Equations

(i) The Volume of a solid generated by the revolution about the x-axis, of the area bounded by the curve y=f(t), y=ø(t), the x-axis and the ordinates, where t=a, t=b is $\int_a^b \pi y^2 \frac{dx}{dt}\, dt$

(ii) The Volume of a solid generated by the revolution about the x-axis, of the area bounded by the curve x=f(t), y=ø(t), the y-axis and the abscissa at the points, where t=a, t=b is $\int_a^b \pi y^2 \frac{dx}{dt}\, dt$

Volume Between two solids

The Volume of a solid generated by the revolution about the x-axis, of the area bounded by the curves y=f(t), y=ø(t), the x-axis and the ordinates, where x=a, x=b is

$$\int_a^b \pi(y_1^2 - y_2^2)\, dx$$

Where y_1 is the ‘y’ of the upper curve and y_2 that the lower curve

The Volume Formulae for Polar Curves

The volume of the solid generated by the revolution of the area bounded by the curve r=f(θ) and the radii vectors θ=α, θ=β

(i) About the initial line OX(θ=0) is $\int_{\beta}^{\alpha} \frac{2}{3}\pi r^3 \, d\theta$

(ii) About the line OY(θ=$\frac{\pi}{2}$) is $\int_{\beta}^{\alpha} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$

Three Practical Forms of the Surface Formula

(i) Surface Formula for Cartesian equations:- The Curved surface of the solid generated by the revolution about the x-axis of the area bounded by the curve v=f(x), the x-axis and the ordinates x=a, x=b is

$$\int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} \, dx, \qquad where \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(i) **Surface Formula for Parametric equations**:- The Curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve x=f(t),y =$\emptyset(t)$,the x-axis and the ordinates at the point t=a, t=b is

$$\int_{t=a}^{t=b} 2\pi y \frac{ds}{dt}\, dt, \qquad where \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}$$

(ii) **Surface Formula for Polar equations**:- The Curved surface of the solid generated by the revolution about the initial line, of the area bounded by the curve r=f(θ) and the radii vector θ=α, θ=β is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi y \frac{ds}{d\theta}\, d\theta, \qquad where \quad \frac{ds}{d\theta}$$

$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad and\ y = r\sin\theta$$

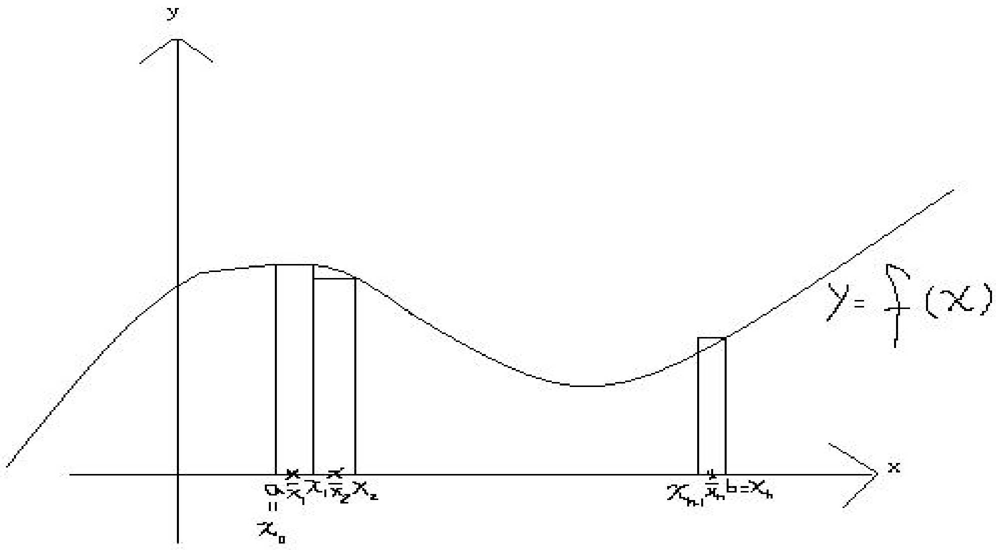
Multiple Integral

Double Integrals over Rectangles

1. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $a = x_0 < x_1 < \cdots < x_n = b$
   Then P is called a partition of [a , b]
2. Define $\Delta x_i = x_i - x_{i-1}$, $i = 1,2,\ldots,n$
3. $|P| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ The norm of P
4. Choose $\bar{x}_i \in [x_{i-1}, x_i]$, $i = 1,2,\ldots n$, $\bar{x}_i$ is called a sample point.

5. The definite integral of f on [a , b]

$$\int_a^b f(x)dx = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i = \text{The area of } S$$



$$S = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

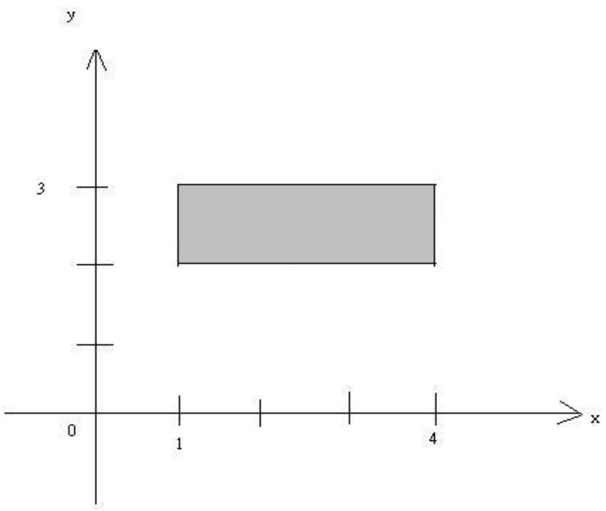
6. A closed rectangle R

$$R = [a, b] \times [c, d] = \{(x, y) \in R^2 \mid a \le x \le b, c \le y \le d\}$$

Example: Rectangle

1. $[0,1] \times [2,4] = \{(x, y) \in R^2 \mid 0 \le x \le 1, 2 \le y \le 4\}$

2. $[1,2] \times [2,$



Evaluation of Double Integral

(i) When $x_1 = \emptyset_1(y)\ and\ x_2 = \emptyset_2(y)$ are the functions of y and y_1, y_2 are constants. Then the double integral of the function f(x,y) over the region R is defined as

$$\iint_R f(x,y)dx\,dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1=\emptyset_1(y)}^{x_2=\emptyset_2(y)} f(x,y)dx}\ dy}$$

(ii) When $y_1 = \emptyset_1(x)\ and\ y_2 = \emptyset_2(x)$ are the functions of x and x_1, x_2 are constants. Then the double integral of the function f(x, y) over the region R is defined as

$$\iint_R f(x,y)dx\,dy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1=\emptyset_1(x)}^{y_2=\emptyset_2(x)} f(x,y)dy}\ dx}$$

(i) When x_1, x_2 *and* y_1, y_2 *are constant* Then the double integral of the function f(x,y) over the region R is defined as

$$\iint_R f(x,y)dx\,dy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)dx} dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)dy} dx}$$

In this case the order of integration is immaterial.

Evaluation of Double Integral in Polar Co-ordinates

The double integral bounded by the straight lines $\theta = \theta_1, \theta = \theta_2$ and the curves $r = r_1, r = r_2$ is defined as

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)dr\,d\theta$$

In polar co-ordinates first we have to integrate w.r.t. r and than w.r.t. the constant limits θ

ASSIGNMENT

Evaluate

(i) $\int_0^3 \int_1^4 x^2 y\,dx\,dy$ (ii) $\int_0^3 \int_1^4 x^2 y\,dy\,dx$

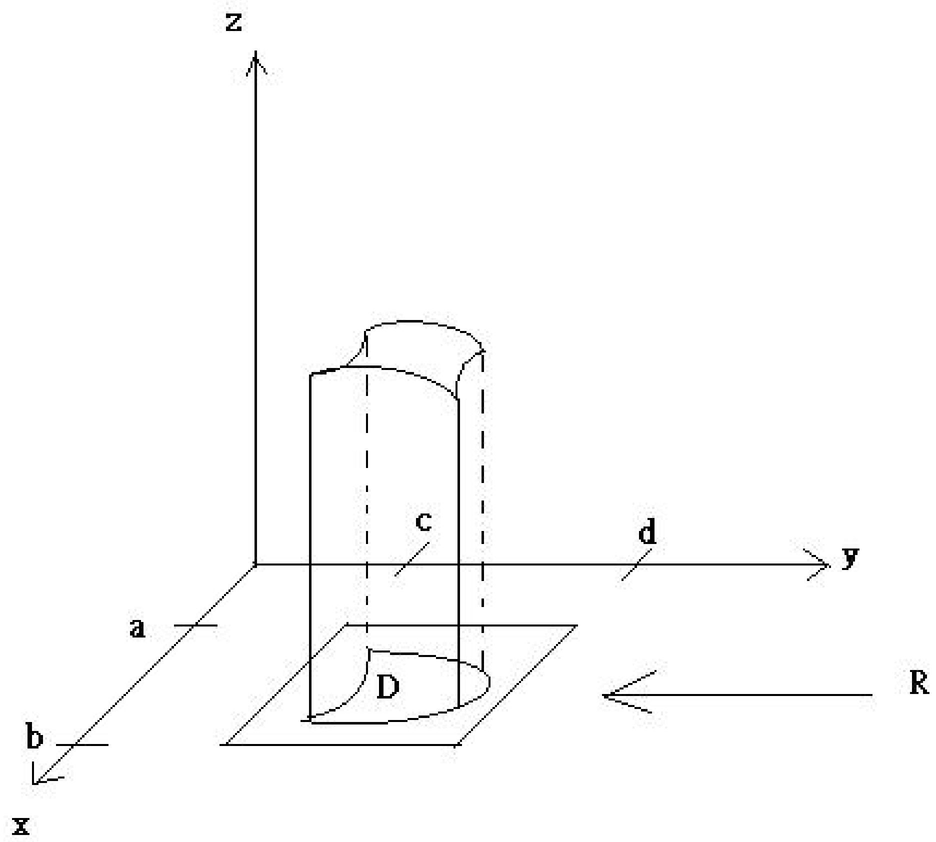
(iii) $\int_0^4 \int_0^8 \frac{1}{4}(64 - 8x + y^2)\,dy\,dx$

(iv) $\int_0^8 \int_0^4 \frac{1}{4}(64 - 8x + y^2)\,dx\,dy$

(v) $\int_0^3 \int_1^2 (3x + 2y)\,dx\,dy$

DOUBLE INTEGRAL OVER GENERAL REGIONS

Let D be a bounded region and $D \subset R$, f is a function defined on D. Define a new function



$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 \text{ if } (x,y) \text{ is in R but not in D} \end{cases}$$

Definition :

1. The double integral of f over D is

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

2. A plane region D is said to be of type I if

$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$,

where g_1, g_2 are two continuous function.

3. A plane region D is said to be of type II if

$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$,

where h_1, h_2 are two continuous function.

Example:

1. $D_1 = \{(x,y) \mid 0 \le x \le \pi, \sin x \le y \le 1\}$, Type I

2. $D_2 = \{(x,y) \mid -1 \le y \le 1, 2y^2 \le x \le 1+y^2\}$, Type II

Properties:

1. If f is continuous on a type I region D such that

   $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

   then $\iint_D f(x,y)\,dA = \int_a^b \int_{g_{1(x)}}^{g_2(x)} f(x,y)\,dy\,dx$

2. If f is continuous on a type II region D then

   $\iint_D f(x,y)\,dA = \int_c^d \int_{h_{1(y)}}^{h_2(y)} f(x,y)\,dx\,dy$

   where $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

Example:

1. Evaluate $\iint_D (x+3y)\,dA$

Where $D=\{(x,y)\mid -1\le x\le 1,\ 2x^2\le y\le 1+x^2\}$

Ans:

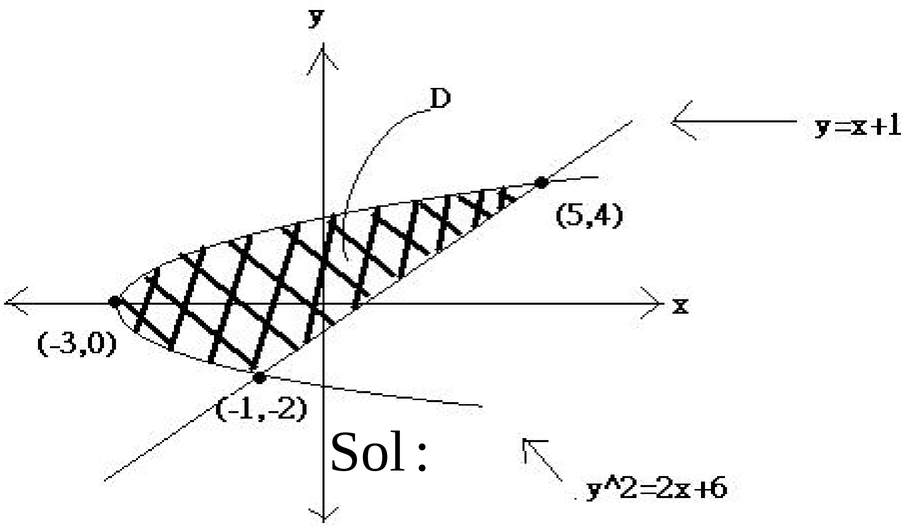
$$\iint_D (x+3y)\,dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+3y)\,dy\,dx$$

$$=\int_{-1}^{1} x(1+x^2-2x^2)+\frac{3}{2}((1+x^2)^2-(2x^2)^2)dx$$

$$=\int_{-1}^{1} x+x^3-2x^3+\frac{3}{2}+3x^2+\frac{3}{2}x^4-4x^4dx$$

$$=(\frac{1}{2}x^2-\frac{1}{4}x^4+\frac{3}{2}x+x^3-\frac{1}{2}x^5)\Big|_{-1}^{1}=\frac{3}{2}+1-\frac{1}{2}=2$$

2. Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



Sol:

$$D = \{(x, y) \mid -3 \le x \le 5, ? \le y \le \sqrt{2x+6}\}$$

$$= \{(x, y) \mid \frac{y^2 - 6}{2} \le x \le y + 1, -2 \le y \le 4\}$$

$$\iint_D xy\,dA = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = 36$$

3. Find the volume of the tetrahedron bounded by the planes

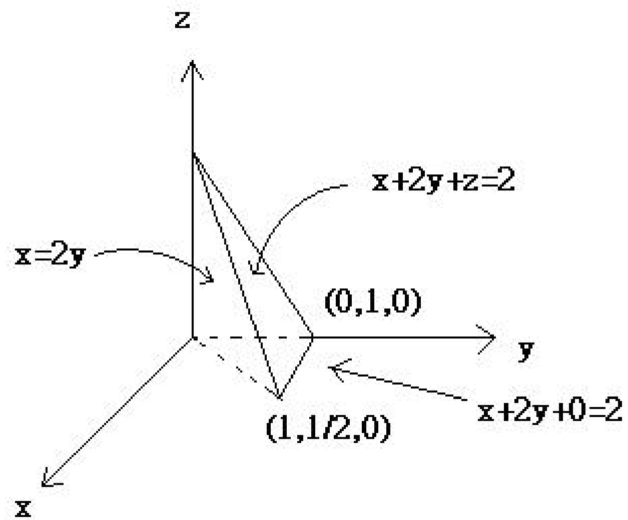
$x = 2y, x = 0, z = 0$ and $x + 2y + z = 2$

Sol :

$$D = \{(x, y) \mid 0 \le x \le 1, \frac{x}{2} \le y \le \frac{2-x}{2}\}$$

$$V = \iint_D 2 - x - 2y\,dA = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2 - x - 2y)\,dy\,dx$$

$$= \frac{1}{3}$$



4. Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$; $\frac{1}{2}(1-\cos 1)$

5. Evaluate $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$

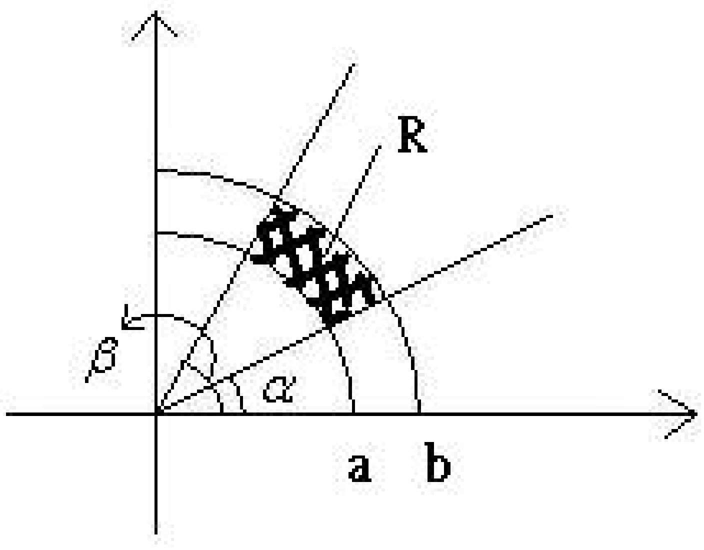
Sol :

$$D = \{(x, y) \mid 0 \le x \le 1, x \le y \le 1\}$$

$$= \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

$$\text{Consider } R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta$$



Polar rectangle

Example：

1. $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

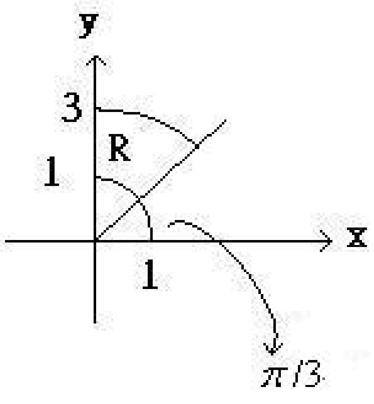
2. $R = \{(r,\theta) \mid 1 \le r \le 3,\ 0 \le \theta \le \pi\}$

3. $R = \{(r,\theta) \mid 1 \le r \le 3,\ \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$

The area of R is $A(R) = (\pi \cdot 3^2 - \pi \cdot 1^2)\dfrac{\frac{\pi}{2} - \frac{\pi}{3}}{2\pi}$

$$= \frac{1}{2}(3^2 - 1^2)\cdot(\frac{\pi}{2} - \frac{\pi}{3})$$

$$= \frac{2}{3}\pi$$



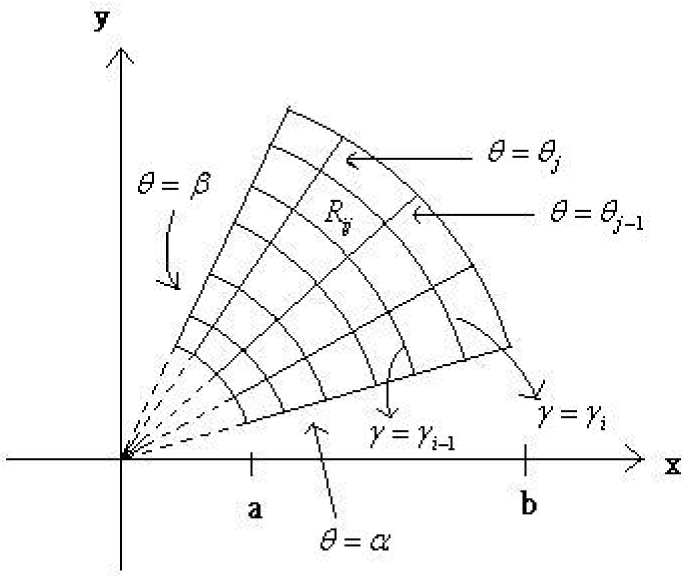
$$4.\ R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

The area of R_{ij} - ΔA_{ij} is

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j$$

$$= r_i^* \Delta r_i \Delta\theta_j$$

Where $\Delta r_i = r_i - r_{i-1},\ \Delta\theta_j = \theta_j - \theta_{j-1,}\ i = 1,\cdots m;\ j = 1,\cdots n$



The Riemanu sum of f on R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

$$\to \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Properties

1. Let $R = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ be a polar rectangle and $0 \le \beta - \alpha \le 2\pi$ If f is continuous on R, then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

2. Let $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ be a polor region. If f is continuous on D then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

NPTEL LINKS FOR REFERENCE

Double integral over rectangular domain.	http://nptel.ac.in/courses/122101003/40

Change of order of integration

In a double integral, if the limits of the Integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of integration help in fixing the new limit of integration.

- change of order of integration
-
-
- https://www.youtube.com/watch?v=BRFUvco07uU

Triple Integrals

The Triple integral of the function f(x,y,z) over the region V is defined as $\iiint_V f(x,y,z)dV$

For purpose of evaluation, it can be expressed as the repeated integral

$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)dx\,dy\,dz$ *The order of integration depending upon the limits*

Let x_1,x_2 be function of y,z; y_1,y_2 be function of z and z_1, z_2 be constant, i.e,

Let $x_1=f_1(y,z)$, $x_2=f_2(y,z)$, $y_1=\emptyset_1(z)$, $y_2=\emptyset_2(z)$ and $z_1=a$, $z_2=b$

Then the Triple integral is evaluated as:-

$$\int_{z_1=a}^{z_2=b}\left[\int_{y_1=\emptyset_1(z)}^{y_2=\emptyset_2(z)}\left[\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\,dx\right]dy\right]dz$$

NPTEL LINK FOR REFERENCE

- Triple integral
- http://nptel.ac.in/courses/122101003/41
-

Change of Variables

Quite often, the evaluation of a double or triple integral is greatly simplified by a suit able change of variables:-

Let the variable x,y in the double integral $\iint_R f(x,y)dx\,dy$ be changed to u,v by means of relations

x=$x=\emptyset(u,v)=\varphi(u,v)$, then the double integral is transformed into $\iint_R f\{\emptyset(u,v),\varphi(u,v)\}|J|du\,dv$,

where $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$

Is the Jacobian of transformation from (x,y) to (u,v) co-ordinates and R' is the region in the uv-plane which corresponds to the region R in the xy-plane

(i) To change cartesian co-ordinates (x,y) to polar co-ordinates(r, θ)

Here we have x=r cosθ, y=r sin θ so that $x^2+y^2=r^2$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}cos\theta & -rsin\theta\\ sin\theta & rcos\theta\end{vmatrix}=r(cos^2\theta+sin^2\theta)=\ r$$

$$\iint_R f(x,y)dx\,dy=\iint_R f(r\,cos\theta,rsin\theta)r\,dr\,d\theta$$

$$i.e;replace\ x\ by\ r\ cos\theta, y\ by\ r\ sin\theta\ \ and\ dx\ dy\ r\ dr\ d\theta$$

(ii) To change the Cartesian co-ordinates(x,y,z) to spherical polar co-ordinates (r, $\theta, \emptyset$)

Here we have

x=r sin $\theta\ cos\emptyset$

y= r sin $\theta\ cos\emptyset$

z=r cos θ

so that $x^2+y^2=r^2$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta cos\emptyset & rcos\theta cos\emptyset & -rsin\theta sin\emptyset \\ sin\theta sin\emptyset & rcos\theta sin\emptyset & rcos\emptyset\ sin\theta \\ cos\theta & -rsin\theta & 0 \end{vmatrix} = r^2 sin\theta$$

$$\iiint_V f(x,y,z)dx\ dy\ dz = \iiint_V f(rsin\theta\ cos\emptyset, rsin\theta\ sin\emptyset, rcos\theta)r^2 sin\theta\ dr\ d\theta\ d\emptyset$$

Note. Equation of the sphere $x^2+y^2+z^2=a^2$ in spherical polar co-ordinates is r=a

(i) If the region of integration is the whole sphere, then

$0 \leq r \leq a, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \emptyset \leq 2\pi$, then

(ii) If the region of the integration is the positive octant

$$0 \leq r \leq a, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \emptyset \leq \frac{\pi}{2}$$

(iii)To change Cartesian co-ordinates (x,y,z) to cylindrical co-ordinates (r, $\emptyset, z$)

Here we have

x=r $cos\emptyset$

y= r sin∅

z= z

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \emptyset} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \emptyset} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \emptyset} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} cos\emptyset & -rsin\emptyset & 0 \\ sin\emptyset & rcos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r^2(cos^2\emptyset + sin^2\emptyset) = r$$

$$\iiint_V f(rcos\emptyset, rsin\emptyset, z)r\ dr\ d\emptyset\ dz$$

$Note - for\ the\ cylinder$ $x^2+y^2+z^2=a^2$, z=0,z=h, the limits of the integration are

$0 \leq r \leq a, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \emptyset \leq h$

If the region of integration is a cylinder (or cone), change the problem to cylindrical polar co-ordinates.

NPTEL LINKS FOR REFERENCE

- Change of variable in double & triple integral.
-
- http://nptel.ac.in/courses/122104017/28
- http://nptel.ac.in/courses/122101003/42
-

Area by the Double integration

(a) Cartesian co-ordinates:- The area A of the region bounded by the curves $y=f_1(x)$, $y=f_2(x)$ and the lines x=a,x=b is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$

The area A of the region bounded by the curves $x=f_1(y)$, $x=f_2(y)$ and the lines y=c, y=d is given by

$$A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx\, dy$$

(a) Polar co-ordinates :- The area A of the region bounded by the curves $f_1(\theta)$, $r=f_2(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$ is given by $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$

Volume as a double integral

$$V = \lim_{\substack{\delta x \to o \\ \delta y \to 0}} \sum z\delta x\delta y = \iint_R z\, dx\, dy$$

Volume as Triple Integral

The Volume V of a three dimensional region is given by $\iiint_V dx\, dy\, dz$

If the region is bounded by $x_1=f_1(y,z)$, $x_2=f_2(y,z)$, $y_1=\emptyset_1(z)$, $y_2=\emptyset_2(z)$ and $z=a$, $z=b$, then

$$V = \int_a^b \int_{\emptyset_1(z)}^{\emptyset_2(z)} \int_{f_1(y,z)}^{f_2(y,z)} dxdy\, dz$$

The order of integration may be changed with a suitable change in the limits of integration.

In cylindrical co-ordinates, we have $V = \iiint_V r\, dr\, d\emptyset\, dz$

In spherical polar co-ordinates, we have $V = \iiint_V r^2 sin\theta\, dr\, d\emptyset\, d\theta$

NPTEL LINKS FOR REFERENCE

- Area and Volume by double integration
- http://www.nptel.ac.in/courses/122101003/40

Volume of Solids of Revolution

If the area R revolves about x-axis, then

$$V = 2\pi \iint_R y\, dy\, dx \qquad (i)$$

If the area R revolves about y-axis, then

$$V = 2\pi \iint_R x dx\, dy \qquad (ii)$$

$$changing\ to\ polar\ co-ordinates\ by\ putting$$

x=r cosθ, y=r sinθ and dy dx=r dr dθ

Formula (i) becomesis a function of n, is called the Gamma Function and is denoted by

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx\ , n > 0$$

$$\text{In Particular,} \Gamma 1 = \int_0^\infty \mathrm{e}^{-\mathrm{x}} \mathrm{dx} = \left|\mathrm{e}^{-\mathrm{x}}\right|_0^\infty = 1$$

Properties of Gamma function

(i) $\Gamma n + 1 = n\Gamma n$

(ii) $\Gamma n + 1 = n!\, when\ n\ is\ a\ positive\ integer$

(iii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

Beta Function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m > 0, n > 0$$

Properties of Beta Function

(i) B(m,n)=B(n,m)

(ii) $B(m,n) = \frac{\Gamma(m)(n)}{\Gamma(m+n)}$

(iii) $\int_0^{\frac{\pi}{2}} sin^p x\, cos^q x\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

$case\ 1\ \ If\ p = q = 0, then$

(iv) $\int_0^{\frac{\pi}{2}} dx = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(1)}$

So $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$Case\ 2\ If\ p = n\ and\ q = 0, then$

(v) $\int_0^{\frac{\pi}{2}} sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} . \frac{\sqrt{\pi}}{2}$

$Case\ 3\ If\ p = 0\ and\ q = n, then$

(vi) $\int_0^{\frac{\pi}{2}} cos^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} . \frac{\sqrt{\pi}}{2}$

Relationship between Beta Gamma Function

we write the product of two factorials as the product of two integrals. To facilitate a change in variables, we take the integrals over a finite range.

$$m!n! = \lim_{a^2 \to \infty} \int_0^{a^2} e^{-u} u^m du \int_0^{a^2} e^{-v} v^n dv, \qquad \begin{array}{l} \Re(m) > -1, \\ \Re(n) > -1. \end{array}$$

Replacing u with x^2 and v with y^2, we obtain

$$m!n! = \lim_{a \to \infty} 4 \int_0^{a} e^{-x^2} x^{2m+1} dx \int_0^{a} e^{-y^2} y^{2n+1} dy.$$

Transforming to polar coordinates gives us

$$m!n! = \lim_{a\to\infty} 4\int_0^a e^{-r^2} r^{2m+2n+3} dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta$$

$$= (m+n+1)! 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta.$$

The definite integral, together with the factor 2, has been named the beta function

$$B(m+1, n+1) \equiv 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta d\theta$$

$$= \frac{m!n!}{(m+n+1)!} = B(n+1, m+1).$$

Equivalently, in terms of the gamma function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$