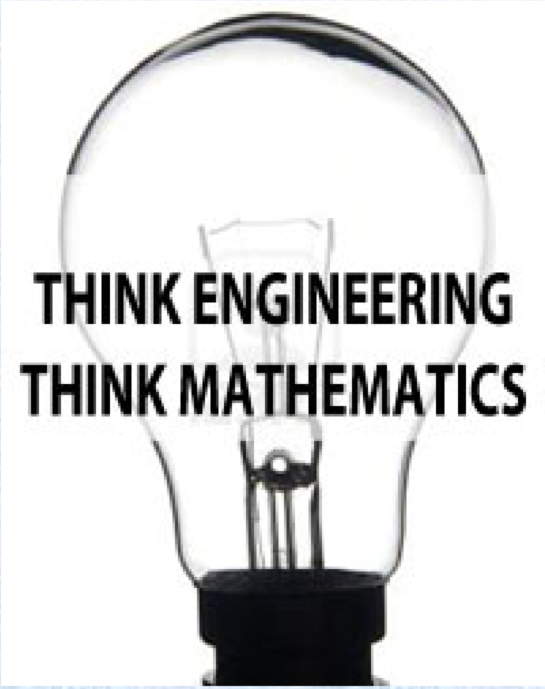
ENGINEERING MATHEMATICS-I SECTION-B



DIFFERENTIAL CALCULUS

- Successive Differentiation
- Leibnitz Theorem and Applications
- Taylor's and Maclaurin's Series
- Curvature
- Asymptotes
- Curve tracing
- Functions of Two or More Variables
- Partial Derivatives of First and Higher Order
- Euler’s Theorem on Homogeneous Functions

- Differentiation of Composite and Implicit unctions
- Jacobians
- Taylor’s Theorem For A Function of Two Variables
- Maxima and Minima of Functions of Two Variables
- Lagrange's Method of Undetermined Multipliers
- Differentiation Under Integral Sign.

E-LEARNING

Topic :Taylor's series.
E-learning: http://nptel.ac.in/courses/122104017/11

Topic :Maclaurin's series.
E-learning: http://nptel.ac.in/courses/122104017/11

Topic : Partial derivatives of first order & its higher order.
E-learning: http://nptel.ac.in/courses/122101003/31

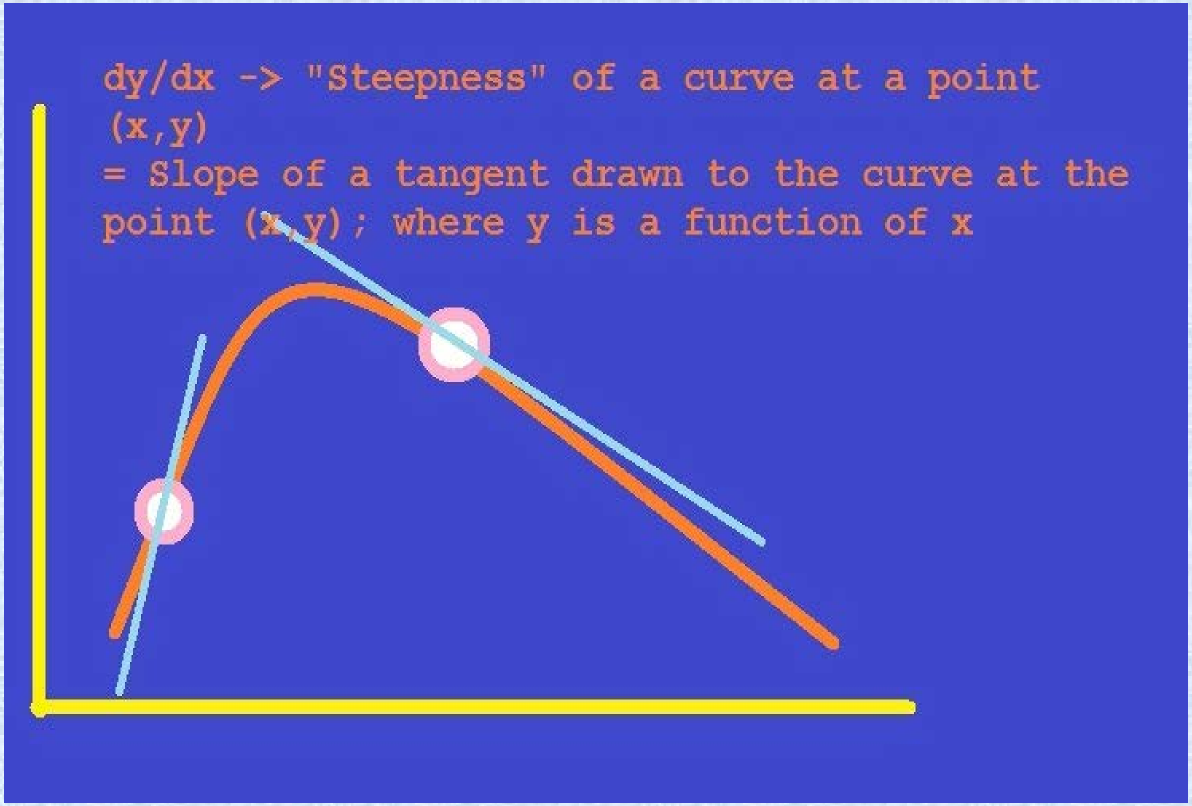
Topic :Euler's theorem on homogeneous functions .
E-learning: www.nptel.ac.in/courses/122101003/downloads/Lecture-31.pdf.

Topic :Total differential,Derivatives of composite and implicit function.
E-learning: http://nptel.ac.in/courses/122101003/32
http://nptel.ac.in/courses/122101003/33

Topic :Maxima and minima of function of two variables.
E-learning: http://nptel.ac.in/courses/122104017/10
http://nptel.ac.in/courses/122101003/37
http://nptel.ac.in/courses/122104017/26

Topic :Lagrange's method of undermined multipliers.
E-learning: http://nptel.ac.in/courses/122104017/27

SUCCESSIVE DIFFERENTIATION



The Process of Differentiating a function again and again is called successive Differentiation.

If y be a function of x, then its successive derivatives are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots \dots \dots \dots \dots \dots \dots, \frac{d^ny}{dx^n}$$

$$y_1, y_2, y_3, \dots \dots \dots \dots \dots \dots \dots \dots \dots, y_n$$

$$y', y'' y''', \dots \dots \dots \dots \dots \dots \dots \dots \dots y^n$$

Example 1. Find the fourth derivative of tan x at $x = \frac{\pi}{4}$

Example 2. $if\ y = Ae^{mx} + Be^{nx}, Prove\ that \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

SOME STANDARD RESULTS

1. n^{th} *derivative of* $x^m = \frac{m!}{(m-n)!} x^{m-n}$ *if* $m \epsilon N, m > n.$
2. n^{th} *derivative of* $(ax+b)^m = m(m-1)(m-2)\ldots\ldots\ldots\ldots\ldots\ldots(m-n+1)(ax+b)^{m-n}\ a^n$ *if* $m \epsilon N, m > n.$
3. *Find the* n^{th} *derivative of* $\frac{1}{ax+b} = \frac{(-1)^n\ n!\ a^n}{(ax-b)^{n+1}}$
4. *Find the* n^{th} *derivative of* $log(ax+b) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
5. n^{th} *derivative of* $a^{mx} = m^n a^{mx} (\log a)^n$
6. n^{th} *derivative of* $e^{mx} = m^n e^{mx}$
7. n^{th} *derivative of* $\sin(ax+b) = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$
8. n^{th} *derivative of* $\cos(ax+b) = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$
9. n^{th} *derivative of* $\mathrm{e}^{\mathrm{ax}} \sin(bx+c) = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx+c+ntan^{-1}\frac{b}{a}\right)$
10. n^{th} *derivative of* $\mathrm{e}^{\mathrm{ax}} \cos(bx+c) = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx+c+ntan^{-1}\frac{b}{a}\right)$

Leibnitz's Theorem

Statement:- if y=uv where u and v are function of x, having derivative of n^{th} order, then

$$y_n = n_{C_0} u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots \ldots \ldots \ldots . . + n_{C_r} u_{n-r} v_r + \cdots \ldots \ldots . . + n_{C_n} u v_n$$

$$where\ suffixes\ denote\ the\ number\ of\ derivatives.$$

Example 1. $If\ y = x^n \log x\ , prove\ that\ y_{n+1} = \frac{n!}{x}$

$Example$ 2.

$If\ y = \cos\ (m \log x), prove\ that\ x^2 y_{n+2} + (2n+1) x y_{n+1} + (m^2 + n^2) y_n = 0$

LINK FOR REFERENCE

- Leibnitz’s Theorem for successive differentiation.
- https://www.youtube.com/watch?v=67uJGwsZz-Q

TAYLOR AND MACLAURIN'S SERIES

- The Taylor's series is named after the English mathematician Brook Taylor (1685–1731).

- The Maclaurin's series is named for the Scottish mathematician Colin Maclaurin (1698–1746).

- This is despite the fact that the Maclaurin's series is really just a special case of the Taylor's series.

APPLICATIONS OF TAYLOR'S AND MACLAURIN'S SERIES

- Expressing the complicated functions in terms of simple polynomials.
- Complicated functions can be made smooth.
- Differentiation of the such functions can be done as often as we please.
- In the field of Ordinary Differential Equations when finding series solution to Differential Equations.
- In the study of Partial Differential Equations.

GENERAL TAYLOR'S SERIES

(I) Expressing f(x + h) in ascending integral powers of h.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots\cdots$$

provided that all derivatives of f(x) are continuous and exist in the interval [x   x+h]

(II) Expressing f(x) in ascending integral powers of (x – a)

$$f(x) = f\big(a + (x-a)\big)$$

$$= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

GUIDELINES FOR FINDING TAYLOR SERIES

Expanding f(x) about x = a

Differentiate f(x) several times

Evaluate each derivative at $x = a$

Evaluate $f(a), f'(a), f''(a)$

Substitute the above values in

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

Example:

Find the Taylor series for f(x) = sin x at c = π/4

f(x) = sin x $$f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

f '(x) = cos x $$f'\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

f ''(x) = - sin x $$f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

f '''(x) = - cos x $$f'''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$f^{(4)}(x)$ = sinx $$f^{(4)}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

Cont….

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-c)^n}{n!}=f(c)+f'(c)(x-c)+.....\frac{f^{(n)}(c)}{n!}(x-c)^n+....$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2\cdot 2!}(x-\frac{\pi}{4})^2-\frac{\sqrt{2}}{2\cdot 3!}(x-\frac{\pi}{4})^3+\frac{\sqrt{2}}{2\cdot 4!}(x-\frac{\pi}{4})^4....$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-c)^n}{n!}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{2\cdot 2!}(x-\frac{\pi}{4})^2.....\frac{\sqrt{2}}{2n!}(x-\frac{\pi}{4})^n+....$$

$$=\frac{\sqrt{2}}{2}\left[1+(x-\frac{\pi}{4})-\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}-\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+\frac{\left(x-\frac{\pi}{4}\right)^4}{n!}+....\right]$$

MACLAURIN'S SERIES

The Maclaurin's series is simply the Taylor's series about the point x = 0

It is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots$$

Find the Maclaurin's series for f(x) = ln(x² + 1)

$$f(x) = \ln(x^2 + 1) \qquad f(0) = 0$$

$$f'(x) = \frac{2x}{1 + x^2} \qquad f'(0) = 0$$

$$f''(x) = \frac{2 - 2x^2}{(x^2 + 1)^2} \qquad f''(0) = 2$$

$$f'''(x) = \frac{4x(x^2 - 3)}{(x^2 + 1)^3} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} \qquad f^{(4)}(0) = -12$$

$$f^{(5)}(x) = \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} \qquad f^{(5)}(0) = 0$$

Cont….

$$f^{(5)}(x)=\frac{48x(x^4-10x^2+5)}{(x^2+1)^5}$$

$$f^{(6)}(x)=\frac{-240(5x^6-15x^4+15x^2-1)}{(x^2+1)^6}$$

$f(0)=0$ $\quad f'''(0)=0$

$f'(0)=0$ $\quad f^{(4)}(0)=-12$

$f''(0)=2$ $\quad f^{(5)}(0)=0$

$f^{(6)}(0)=240$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}x^n}{n!}=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2.....\frac{f^{(n)}(0)}{n!}x^n+....$$

$$=0+0+\frac{2}{2!}x^2+\frac{0}{3!}x^3+\frac{-12}{4!}x^4+\frac{0}{5!}x^5+\frac{240}{6!}x^6.........$$

$$=x^2-\frac{x^4}{2}x^4+\frac{x^6}{3}.........$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}$$

Find the Taylor series for $f(x) = e^{-2x}$ at $c = 0$

$f(x) = e^{-2x}$ $\quad$ $f(0) = 1$

$f'(x) = -2e^{-2x}$ $\quad$ $f'(0) = -2$

$f''(x) = 4e^{-2x}$ $\quad$ $f''(0) = 4$

$f'''(x) = -8e^{-2x}$ $\quad$ $f'''(0) = -8$

$f^{(4)}(x) = 16e^{-2x}$ $\quad$ $f^{(4)}(0) = 16$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

$$= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \ldots\ldots \frac{2^n x^n}{n!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

MACLAURIN'S SERIES

We defined:

- *the nth Maclaurin polynomial for a function as*

$$\sum_{k=0}^{n} \frac{f^{k}(0)}{k!}x^{k} = f(0) + f^{/}(0)x + \frac{f^{//}(0)}{2!}x^{2} + \ldots + \frac{f^{n}(0)}{n!}x^{n}$$

- *the nth Taylor polynomial for f about $x = x_0$ as*

$$\sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}(x - x_0)^{k} = f(x_0) + f^{/}(x_0)(x - x_0) + \frac{f^{//}(x_0)}{2!}(x - x_0)^{2} + \ldots + \frac{f^{n}(x_0)}{n!}(x - x_0)^{n}$$

Example

Derive the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Maclaurin series is simply the Taylor series about the point x=0

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

DERIVATION FOR MACLAURIN SERIES FOR

$$e^x$$

DERIVATION (CONT.)

Since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^n(x) = e^x$ and

$f^n(0) = e^0 = 1$

the Maclaurin series is then

$$f(h) = (e^0) + (e^0)h + \frac{(e^0)}{2!}h^2 + \frac{(e^0)}{3!}h^3 ...$$

$$= 1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 ...$$

So,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Find the Maclaurin polynomial for f(x) = x cos x

We find the Maclaurin polynomial cos x and multiply by x

f(x) = cos x	**f(0) = 1**
f ’(x) = -sin x	**f ’(0) = 0**
f ”(x) = - cos x	**f ”(0) = - 1**
f ”’(x) = sin x	**f ”’(0) = 0**
$f^{(4)}(x)$ = cos x	**$f^{(4)}(0)$ = 1**

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots\ldots$$

$$= 1 + 0 - \frac{4x^2}{2!} - 0 + .\frac{x^4}{4!}\ldots\ldots \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\ldots\ldots$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!}\ldots\ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

Find the Maclaurin polynomial for f(x) = sin 3x

We find the Maclaurin polynomial sin x and replace x by 3x

f(x) = sin x	**f(0) = 0**
f '(x) = cos x	**f '(0) = 1**
f ''(x) = - sin x	**f ''(0) = 0**
f '''(x) = - cos x	**f '''(0) = -1**
$f^{(4)}$(x) = sin x	**$f^{(4)}$(0) = 0**

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots\ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0\ldots\ldots \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\ldots\ldots$$

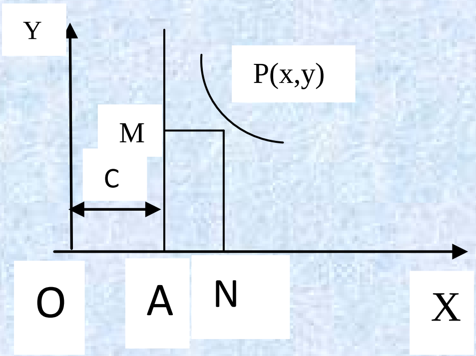
$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}\ldots\ldots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

- Taylor’s & Maclaurin’s Theorem for one variable.
- http://nptel.ac.in/courses/122104017/11
- http://www.creativeworld9.com/2011/02/iit-guest-lecture-mathematics-iii-video.html

ASYMPTOTES

Definition: An **asymptote** of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In other words..

A Straight line at a finite distance from the origin, is said to be an asymptote of an **infinite branch of a curve**, if the perpendicular distance of a point P on that branch from the straight line tends to zero as P tends to infinity along the branch of the curve.



A Curve With Finite Branches	A Curve With Infinite Branches
Ellipse :	Hyperbola :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

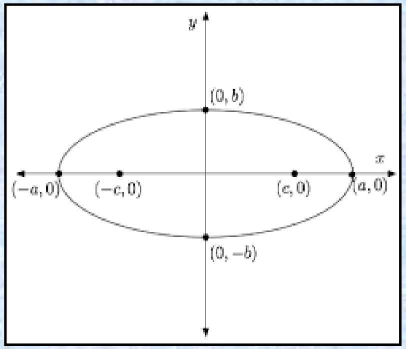
Its two branches are

y= $b\sqrt{1-\frac{x^2}{a^2}}$ **and y= -b**$\sqrt{1-\frac{x^2}{a^2}}$

(upper half) **(lower half)**

(Both branches lie within x=a,x= -a ,y=b ,

y=-b.)

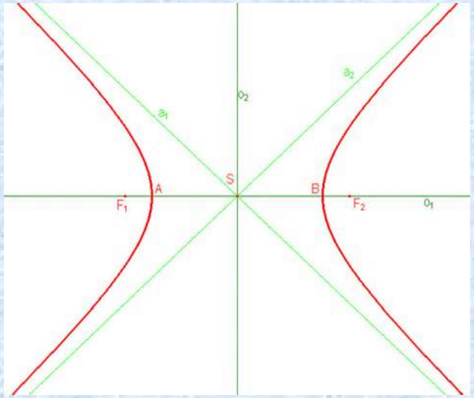


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

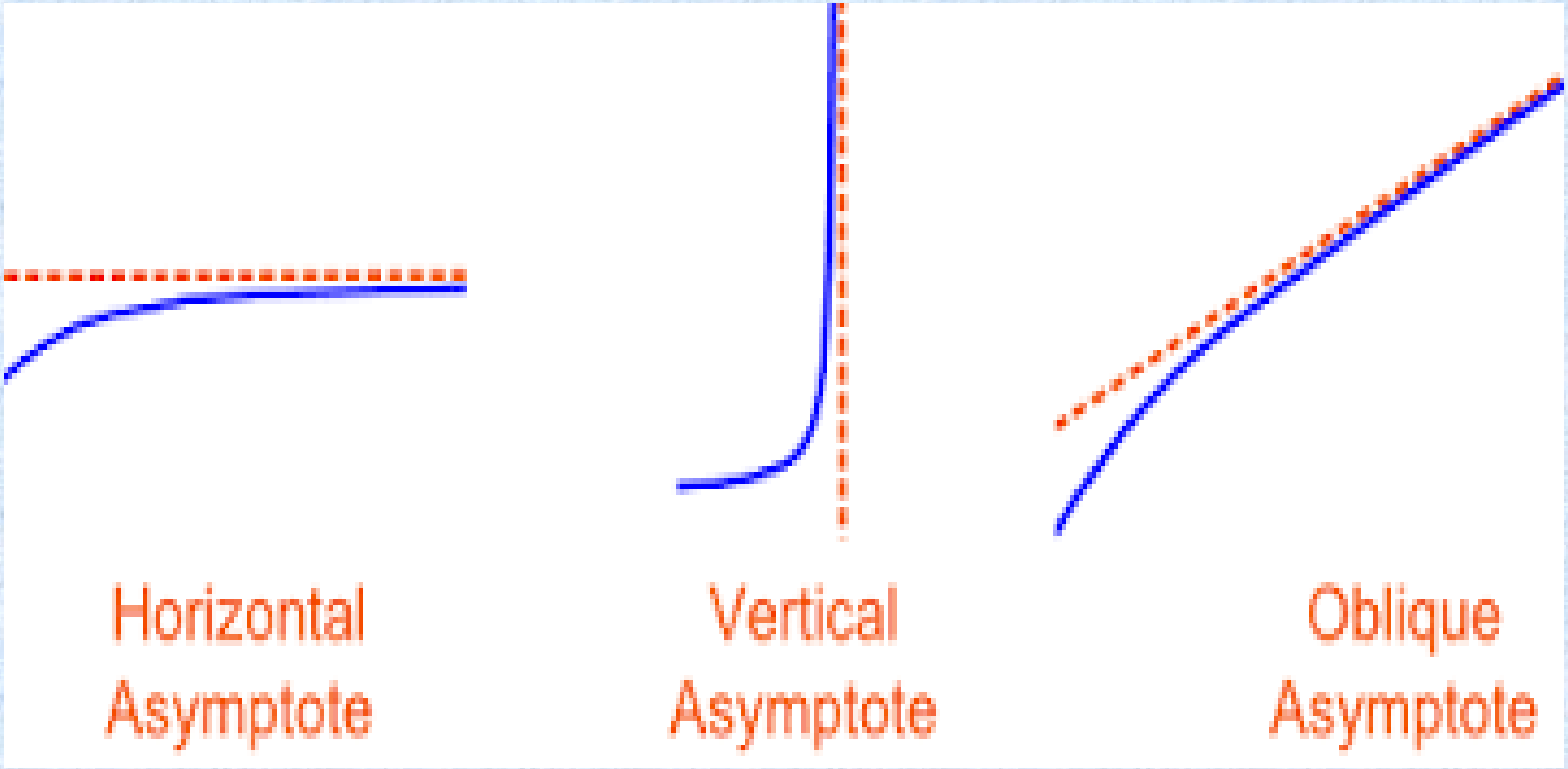
Its infinite branches are

$y=\frac{b}{a}\sqrt{x^2-a^2},\quad y=-\frac{b}{a}\sqrt{x^2-a^2}$

(Here $y\to\pm\infty$ **as** $x\to\pm\infty$**)**



KINDS OF ASYMPTOTES



ASYMPTOTE PARALLEL TO AXES

Asymptote Parallel to x-axis

Rule to find the asymptote || to X-axis, is to equate to zero the real linear factors in the co-efficient of the highest power of x in the equation of the curve.

Asymptote Parallel to y-axis

Rule to find the asymptote || to Y-axis, is to equate to zero the real linear factors in the co-efficient of the highest power of y in the equation of the curve.

$$Example\ 1. Find\ the\ x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 Asmptote\ Parallel\ to\ axes$$

$$Example\ 6. Find\ the\ x^2y^2 - x^2y - xy^2 + x + y + 1 = 0\ Asmptote\ Parallel\ to\ axes$$

Oblique Asymptote

The equation of straight line y=mx+c is the oblique asymptote to the given curve

WORKING RULE FOR FINDING OBLIQUE ASYMPTOTES OF AN ALGEBRAIC CURVE OF THE NTH DEGREE

1. Find the $\emptyset_n(m)$. This can be obtained by putting x=1, y=m in the highest degree terms of the given equation of the curve.
2. Equate $\emptyset_n(m)$ to zero and solve for m.
   Let its roots be m_1, m_2, m_3,………..
3. Find $\emptyset_{n-1}(m)$ by putting x=1 and y=m in the next lower terms of the equation. Similarly $\emptyset_{n-2}(m)$ may be found out by putting x=1 and y=m in the next lower degree terms in the curve and so on.
4. Find the values of c1,c2,c3,……………….. corresponding to the values m_1, m_2, m_3,………..by using equation $c = \frac{\emptyset_{n-1}(m)}{\emptyset_n'(m)}$
5. Then the required asymptotes are y= m_1x+c_1 ,y= m_2+c_2,……………………..

6. If $\emptyset_n^{'}(m) = 0\ for\ some\ value\ of\ m\ and\ \emptyset_{n-1}(m) \neq 0$corressponding to that value, then there will be no asymptote corresponding to that value of m.
7. If $\emptyset_n^{'}(m) = 0$ and $\emptyset_{n-1}(m) \neq 0$ for some value of m, the value of c are determined by

$$\frac{c^2}{2!}\emptyset_n^{''}(m) + c\emptyset_{n-1}^{'}(m) + \emptyset_{n-2}(m) = 0,$$

And this will determine two value of c and thus we shall have two parallel asymptotes corresponding to this value of m.

$Example\ 1. Find\ the\ asymtote\ of\ the\ curve\ (x-y)^2(x+2y-1)$
$= 3x + y - 7$

$Example\ 2. Find\ all\ the\ asymtote\ of\ the\ following\ curve$

$(i) y^2(x-2a) = x^3 - a^3$

$(ii) y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2x + 2y + 1 = 0$

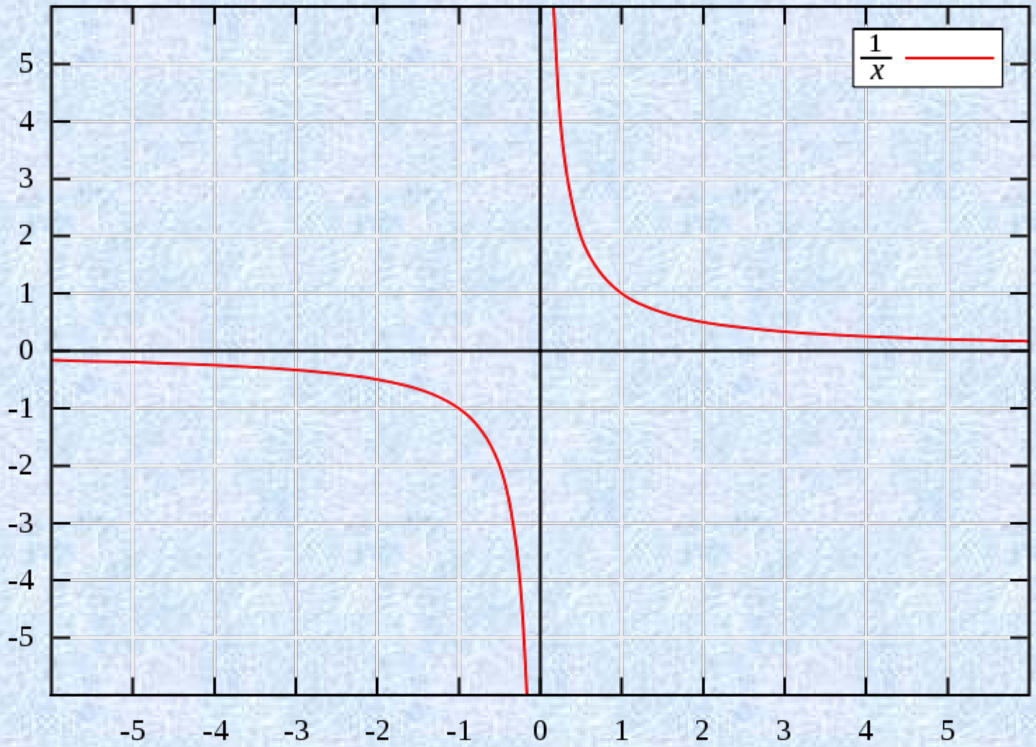
$(iii) y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 0$

$Example\ 3. show\ that\ the\ asymptotes\ of\ the\ curve\ x^2y^2 - a^2(x^2+y^2)$
$- a^3(x+y) + a^4 = 0$

Form the square through two of whose vertices the curve passes.

PICTORIAL EXAMPLE 1

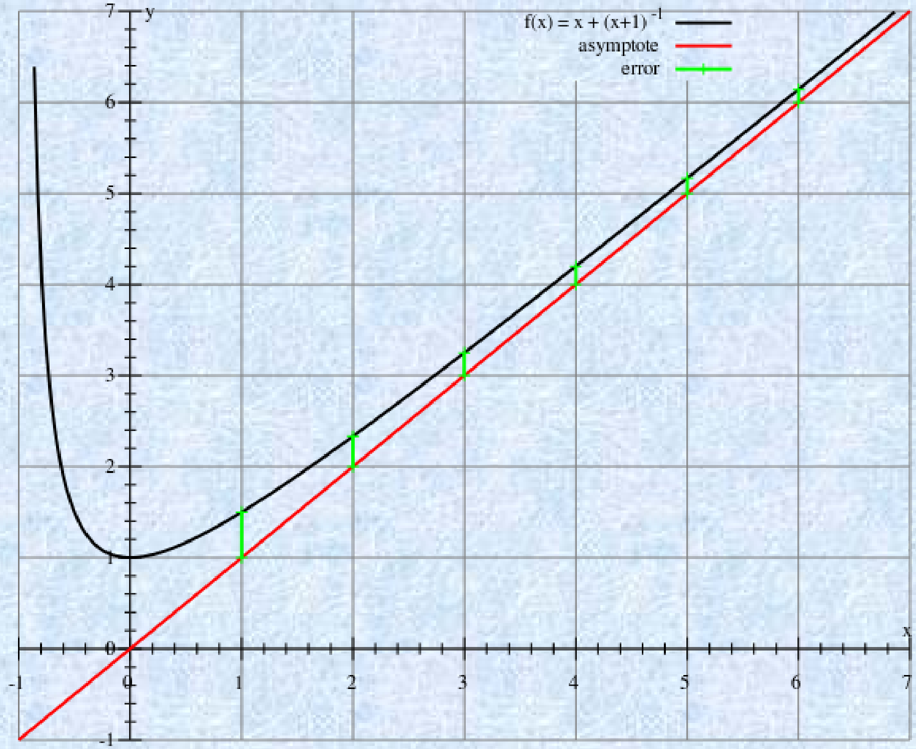
$f(x) = \frac{1}{x}$ **graphed on Cartesian coordinates.**



The *x* and *y*-axes are the asymptotes of the curve.

PICTORIAL EXAMPLE 2

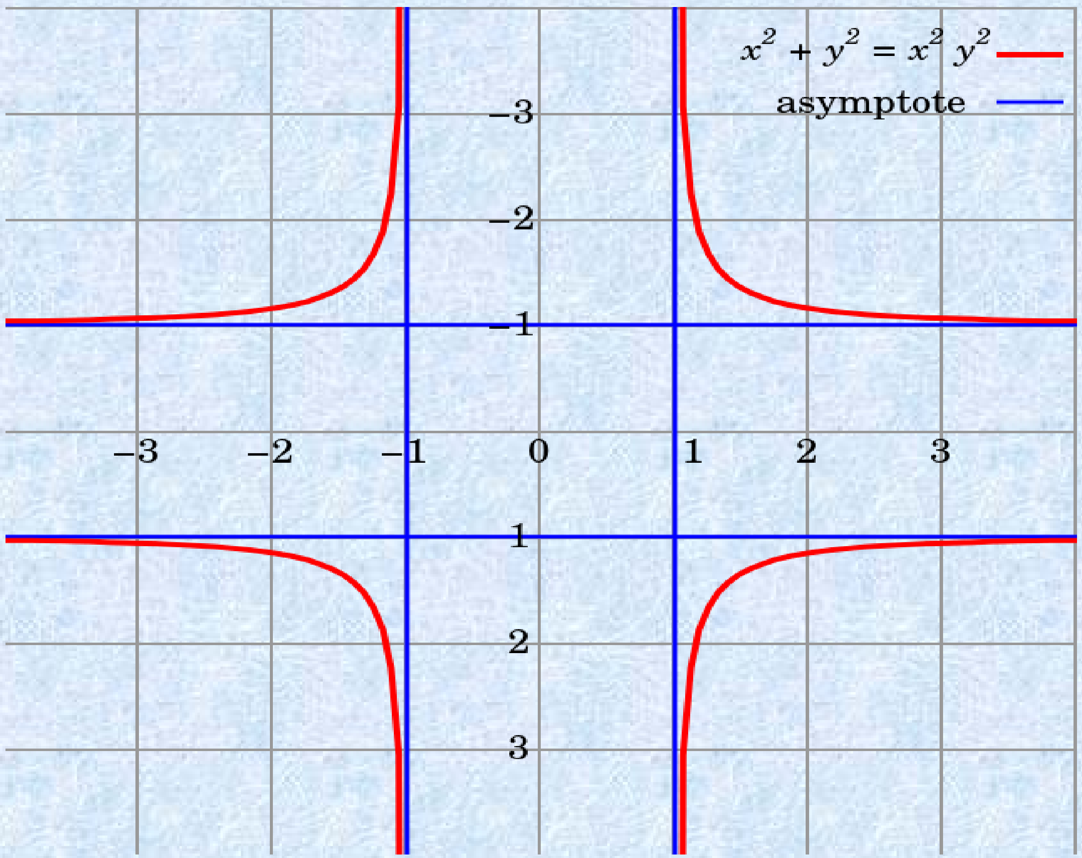
The graph of $f(x) = (x^2 + x + 1)/(x + 1)$



y = x is the Asymptote

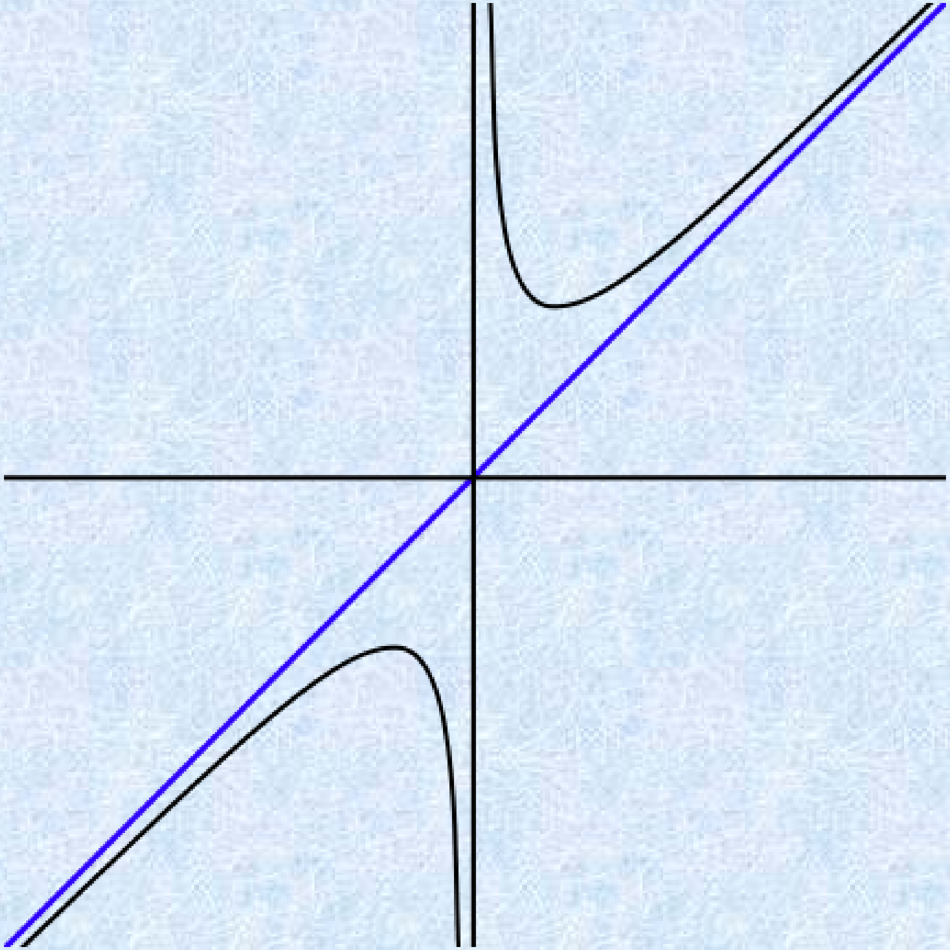
PICTORIAL EXAMPLE 3

The graph of $x^2 + y^2 = (xy)^2$, with 2 horizontal and 2 vertical asymptotes



PICTORIAL EXAMPLE 4

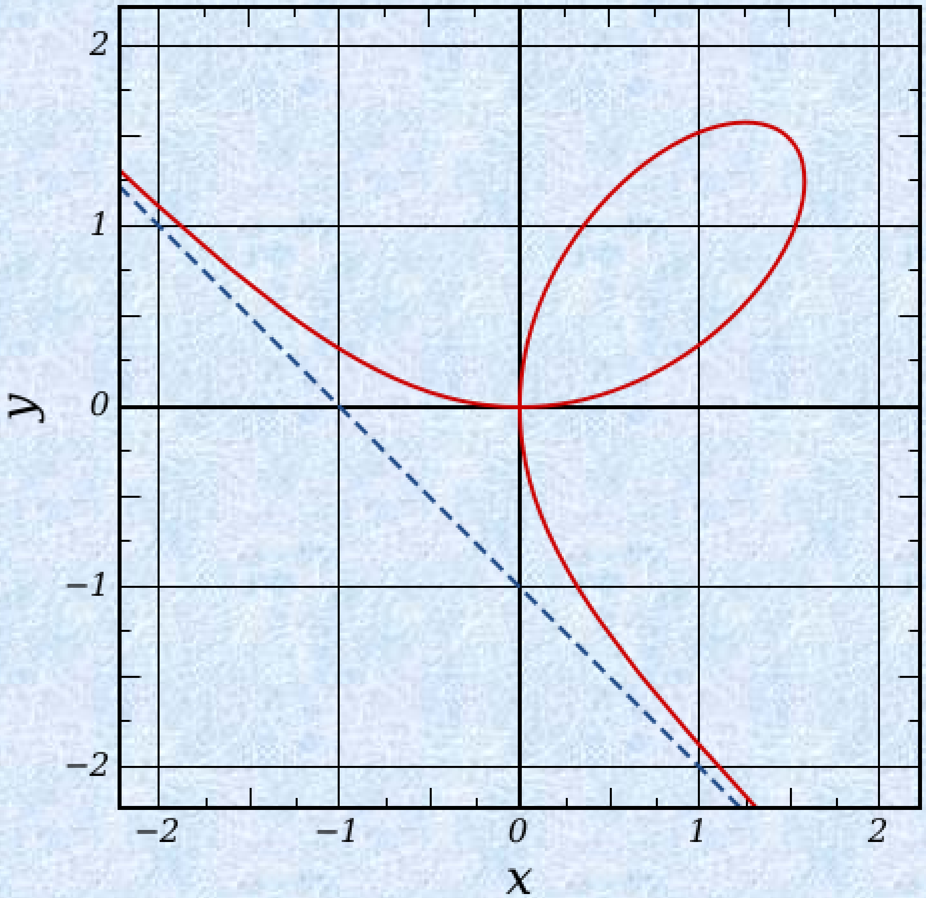
The graph of $f(x) = x + \frac{1}{x}$



The y-axis (x = 0) and the line y = x are both asymptotes

PICTORIAL EXAMPLE 5

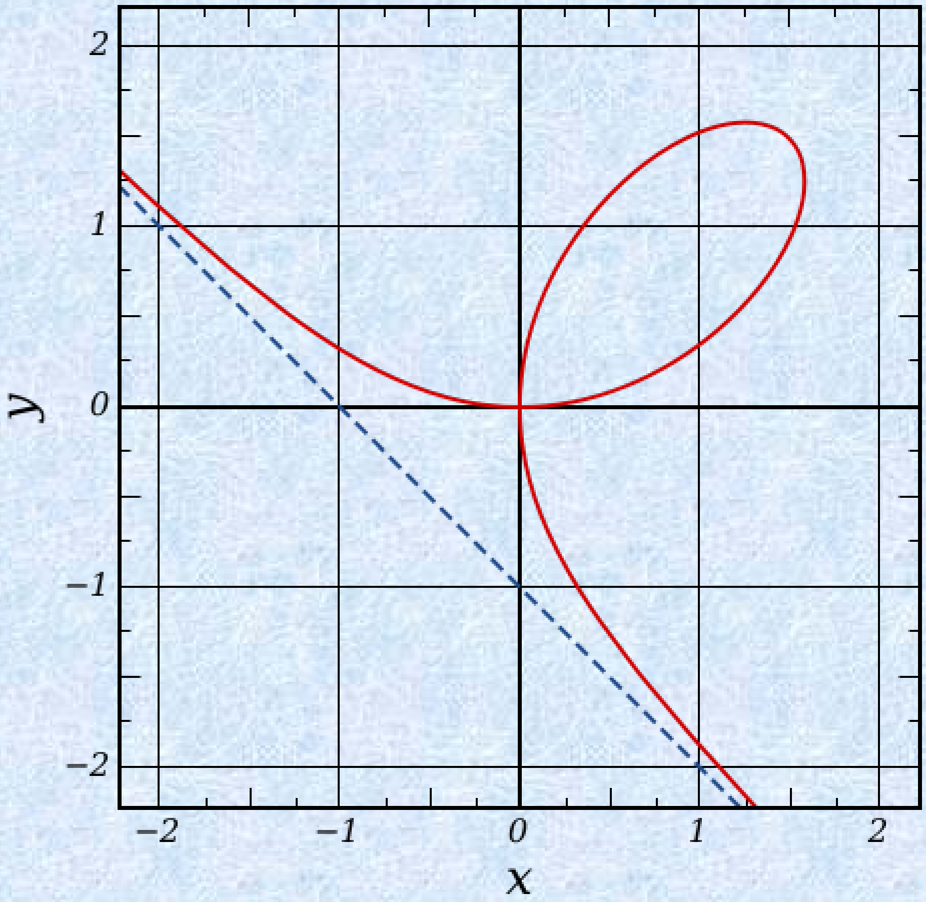
The graph of $x^3 + y^3 = 3axy$



A cubic curve, the folium of Descartes (solid) with a single real asymptote (dashed) given by $x + y + a = 0$.

PICTORIAL EXAMPLE 5

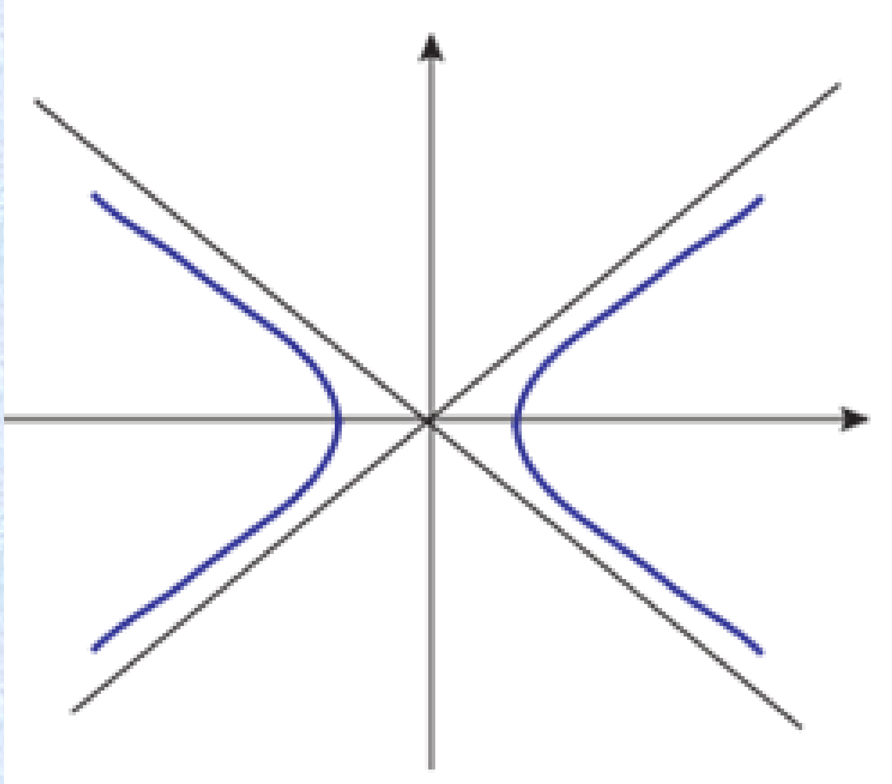
The graph of $x^3 + y^3 = 3axy$



A cubic curve, the folium of Descartes (solid) with a single real asymptote (dashed) given by $x + y + a = 0$.

PICTORIAL EXAMPLE 6

The graph of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Its asymptotes are y = $\pm \frac{b}{a}x$

ASYMPTOTE OF THE POLAR CURVES

$If\ \alpha\ is\ a\ root\ of\ the\ equation\ f(\theta) = 0, then\ \ r\sin(\theta-\alpha) = \frac{1}{f^{'}(\alpha)}\ is\ an\ asympote\ of\ the\ polar\ curve\ \frac{1}{r} = f(\theta)$

Working rule for finding the asymptotes of polar curves.

1. Write down the given equation as $\frac{1}{r} = f(\theta)$
2. Equate $f(\theta)$ to zero and solve for $\theta = \theta_1, \theta_2, \theta_3, \ldots\ldots\ldots\ldots$
3. Find $f^{'}(\theta)$ and calculate $f^{'}(\theta)$ at $\theta = \theta_1, \theta_2, \theta_3, \ldots\ldots\ldots\ldots$
4. Then write asymptote as $r\sin(\theta-\theta_1) = \frac{1}{f^{'}(\theta_1)}, r\sin(\theta-\theta_2) = \frac{1}{f^{'}(\theta_2)}$,……………………..

IMPORTANT FORMULAS

$1. If\ \sin(\theta) = 0, then\ \theta = \frac{\pi}{4}\}$

$2. If cos\theta = 0, then\ \theta = (2n+1)\frac{\pi}{2}$

$3. If sin\theta = sin\alpha\ , then\ \theta = n\pi + (-1)^n \alpha$

$4. If cos\theta = cos\alpha\ , then\ \theta = 2n\pi \pm \alpha$ $\quad n \in \mathrm{I}$

$5. If tan\theta = tan\alpha\ , then\ \theta = n\pi + \alpha$

$6. \sin(n\pi + \theta) = (-1)^n \sin\theta$

$7. \cos(n\pi + \theta) = (-1)^n \cos\theta$

$8. \tan(n\pi + \theta) = tan\ \theta$

$Example\ \ find\ the\ asymptotes\ of\ the\ following\ polar\ curves$

$(i) r = a\tan\theta$

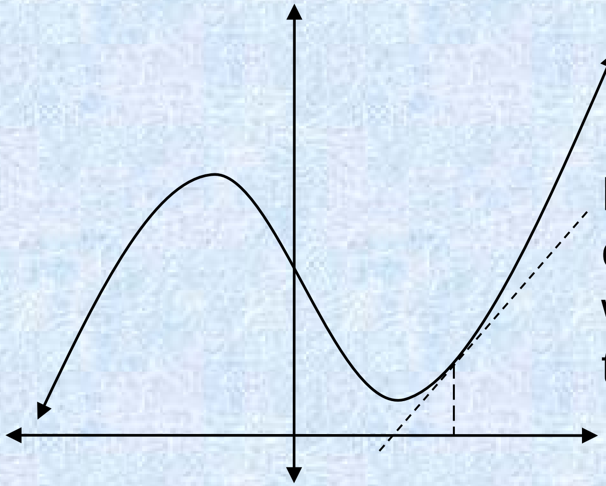
$(ii) r\sin\theta = 2\cos 2\theta$

Derivative of a function

Single-Variable Function

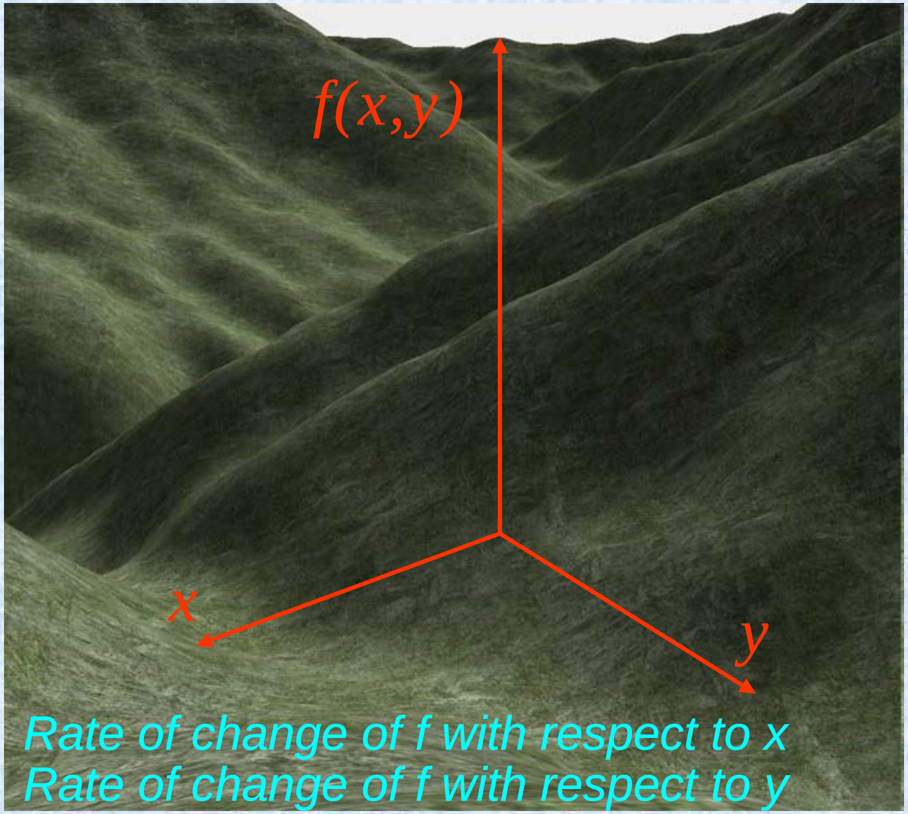
Recall how we find the derivative for a Single Variable function f(x)

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

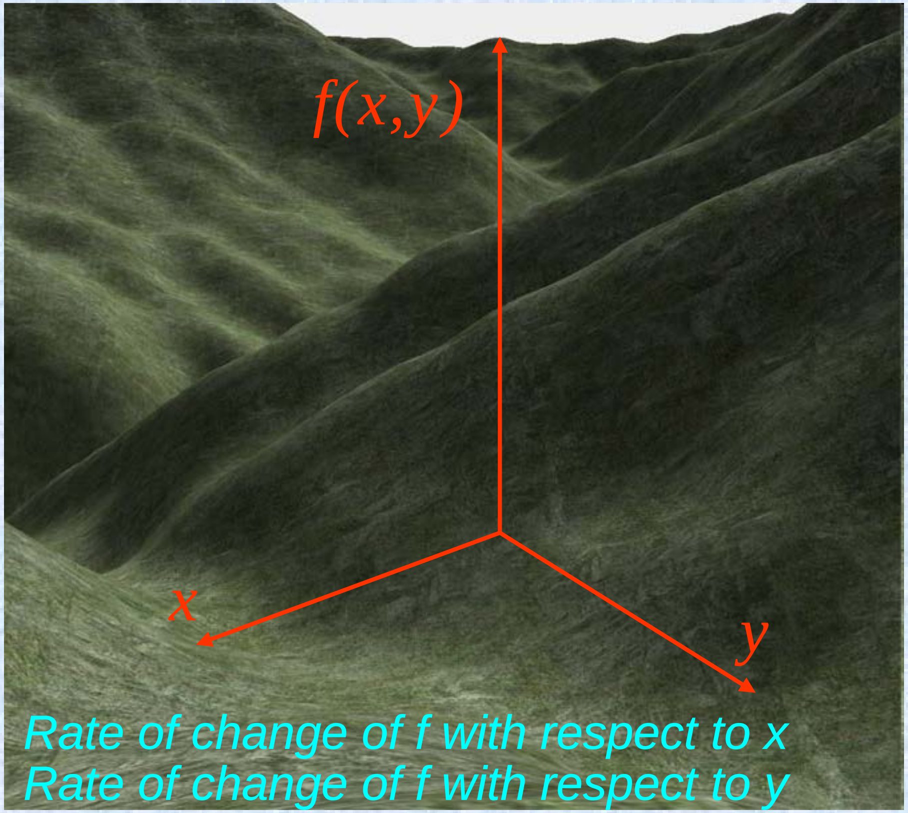


Rate of change of f with respect to x (slope)

Two-Variable Function



Partial derivatives of a function



Partial Derivative of f with respect to x
Partial Derivative of f with respect to y

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Remarks:

- It is called the Partial Derivative because it describes the derivative in one direction.
- Scripted "d", not the regular "d" or "2"
- When differentiate f with respect to x, we treat y as if y were a constant, and vice versa.

Ex: Given $f(x,y) = x^3 - x^2y + xy + 3y^2$

Find $\frac{\partial f}{\partial x}$

HERE: we treat "y" as a constant!!!!

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 - x^2y + xy + 3y^2)$$

$$= \frac{\partial}{\partial x}(x^3) - y\frac{\partial}{\partial x}(x^2) + y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(3y^2)$$

$$= 3x^2 - y(2x) + y(1) + 0$$

$$= 3x^2 - 2xy + y$$

Assignment

If $w = x^2 - xy + y^2 + 2yz + 2z^2 + z,$

find $\dfrac{\partial w}{\partial x}, \quad \dfrac{\partial w}{\partial y}, \quad \text{and} \quad \dfrac{\partial w}{\partial z}.$

Example: A cellular phone company has the following production function for a certain product:

$$p(x, y) = 50x^{2/3}y^{1/3},$$

where p is the number of units produced with x units of labor and y units of capital.

a) Find the number of units produced with 125 units of labor and 64 units of capital.

b) Find the marginal productivities of labor and of capital.

c) Evaluate the marginal productivities at $x = 125$ and $y = 64$.

Higher-Order Derivatives

Single-Variable Function

$$f'(x) = \frac{df}{dx}$$ (derivative)

$$f''(x) = \frac{d^2 f}{dx^2}$$ (2nd derivative)

$$f'''(x) = \frac{d^3 f}{dx^3}$$ (3rd derivative)

Multi-Variable Function

$$f_x = \frac{\partial f}{\partial x}$$

(partial derivative of f wrt x)

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

(2nd partial derivative of f wrt x)

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}$$

(3rd partial derivative of f wrt x)

Ex: Given $f(x,y) = x^3 - x^2y + xy + 3y^2$

We found $f_x = \dfrac{\partial f}{\partial x} = 3x^2 - 2xy + y$

Find f_{xxxx}

Find $\dfrac{\partial^2 f}{\partial y^2}$

Mixed Derivatives

$$f_{xy} = \left(f_x \right)_y = \left(\frac{\partial f}{\partial x} \right)_y = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \left(f_y \right)_x = \left(\frac{\partial f}{\partial y} \right)_x = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

so $f_{xy} = f_{yx}$

Assignment

If $z = f(x,y) = x^2y^3 + x^4y + xe^y,$

find the following partial derivatives:

$$f_x =$$

$$f_{xx} =$$

$$f_{xy} =$$

$$f_y =$$

$$f_{yy} =$$

$$f_{yx} =$$

A Function f(x,y) is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \, \emptyset\left(\frac{y}{x}\right) \, or \; y^n \, \emptyset\left(\frac{x}{y}\right)$

An alternative test for a function f(x,y) to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y)$$

For example, if $f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$, then

$$(i) \; f(x,y) = \frac{x(1+\frac{y}{x})}{\sqrt{x}(1+\sqrt{\frac{y}{x}}} = x^{1/2} \emptyset\left(\frac{y}{x}\right)$$

➡ f(x,y) is a homogeneous function of degree ½ in x and y.

Similarly, a function f(x,y,z) is said to be homogeneous of degree n in the variables x,y,z if

$$f(x,y,z) = x^n \emptyset\left(\frac{y}{z},\frac{z}{x}\right) \quad or \quad y^n (\emptyset) \left(\frac{x}{y},\frac{z}{y}\right) \quad or \quad z^n \emptyset\left(\frac{x}{z},\frac{y}{z}\right)$$

Alternative test is f(tx,ty,tz)= t^n f(x,y,z)

Euler's Theorem on Homogeneous Functions

If u is a homogeneous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Since u is a homogeneous function of degree n in x and y , it can be expressed as $u = x^n\, f\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = nx^{n-1}\, f\left(\frac{y}{x}\right) = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nx^n\, f\left(\frac{y}{x}\right) - x^{n-1}\, y\, f'\left(\frac{y}{x}\right) \qquad (i)$$

$$Also \qquad \frac{\partial u}{\partial y} = x^n\, f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y} = x^{n-1}\, y\, f'\left(\frac{y}{x}\right) \qquad (ii)$$

Adding (i) and (ii), we get $x\frac{\partial u}{\partial x}+\ y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)=nu$

If u is a Homogeneous function of degree n in x and y , then $x^2\frac{\partial^2 u}{\partial x^2}+$ $2xy\ \frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

$$Example\ 1. if\ u\ \ \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}\ ,prove\ that\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\ \frac{\partial^2 u}{\partial x\partial y}$$

$$+\ y^2\frac{\partial^2 u}{\partial y^2}=\ -\ \frac{\sin u\cos 2u}{4\ cos^3 u}$$

Composit functions

$$(i)if\ u=f(x,y)\ where\ x=\emptyset(t),y=\varphi(t)$$

Then u is called a composit function of t and we can find du/dt

$$(ii)\ if\ z=f(x,y)where\ x=\emptyset(u,v),y=\varphi(u,v)$$

Then z is called a composite function of u and v so that we can find $^{\partial z}/_{\partial u}\ \ and\ ^{\partial z}/_{\partial v}$

Cor. 1. If u=f(x,y,z) and x,y,z are function of t, then y is a composite function of t and $\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt} + \frac{\partial u}{\partial z} . \frac{dz}{dt}$

Cor. 2. If z = f(x,y) and x and y are the functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u} \quad ; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v}$$

Cor. 3. If u=f(x,y) where y= $\emptyset(x)$ then since x = $\varphi(x)$, u is a composite function of x

$$\frac{du}{dx} = \frac{\partial u}{\partial x} . \frac{dx}{dx} + \frac{\partial u}{\partial y} . \frac{dy}{dx} \quad => \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$$

Cor. 4. If we are given a implicit function f(x,y) = c , then u=f(x,y) where u=c using cor. 3 , we have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$$

But du/dx=0

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx} = 0 \qquad or \qquad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = - \frac{f_x}{f_y}$$

Hence the differential coefficient of f(x,y) w.r.t x is $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . \frac{dy}{dx}$

Cor 5. If f(x,y) = c, then by cor 4, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$Differentiating\ again\ w.r.t.\ x, we\ get$$

$$\frac{d^2u}{d^2x} = -\frac{f_y\ \frac{d}{dx}\ (f_x) - f_x \frac{d}{dx}\ (f_y)}{f_y^2} = -\frac{f_y\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}\right] - f_x\left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\frac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\frac{f_x}{f_y}\right]}{f_y^2}$$

$$= -\frac{f_{xx} f_y^2 - f_x f_y f_{xy} - f_x f_y f_{xy} - f_{yy} f_x^2}{f_y^3}$$

$$Hence\ \frac{d^2y}{dx^2} = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} - f_{yy} f_x^2}{f_y^3}$$

$Example\ 1. If\ z = 2xy^2 - 3x^2y$ and f increases at the rate of 2 cm per second when it passes through the value $x = 3cm$, show that if y is passing through the value $y = 1$ cm, y must be decreasing at the rate of $2\frac{2}{15}$ cm per second, in order that z shall remain constant.

$Example\ 2.$ if u is a homogeneous function of nth degree in x, y, z, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

$Example\ 3.$ Find $\frac{dy}{dx}$, when

(i) $x^y + y^x = c$ (ii) $(\cos x)^y = (\sin y)^x$

NPTEL LINKS FOR REFERENCE

Partial derivatives	http://nptel.ac.in/courses/122101003/31
Partial derivatves and euler th.	www.**nptel**.ac.in/courses/122101003/downloads/Lecture-31.pdf.

JOCOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u,v with respect to x,y and is denoted by symbol $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Simolarly, if u,v,w be the function of x,y,z, then the Jacobian of u,v,w with respect to x,y,z is

$$J\left(\frac{u,v,w}{x,y,z}\right) \quad or \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of JACOBIANS

1. If u,v are functions of r,s where r, s are functions of x,y, then
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} . \frac{\partial(r,s)}{\partial(x,y)} \qquad [\,chain\ Rule\ for\ Jacobians]$$
2. If J_1 is the Jacobian of u,v, with respect to x,y and J_2 is the Jacobian of x,y with respect to u,v, then J_1J_2 =1 i.e., $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(u,v)} = 1$

$$Example\ 1.\ If\ x = r\ sin\theta\ cos\emptyset\ ,z = r\cos\theta\ ,show\ that\ \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\ \sin\theta$$

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

- A function $f(x,y)$ is said to have a maximum value at $x = a$, $y = b$ if $f(a,b) > f(a+h,b+k)$, for small and independent values of h and k, positive or negative.
- A function $f(x,y)$ is said to have a minimum value at $x = a$, $y = b$ if $f(a,b) < f(a+h,b+k)$, for small and independent values of h and k, positive or negative.

RULE TO FIND THE EXTREME VALUES OF A FUNCTION

Let z = f(x,y) be a function of two variables

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) Solve $\frac{\partial z}{\partial x}$ =0 and $\frac{\partial z}{\partial y}$ =0 simultaneously.

Let (a,b); (c,d)….. Be the solutions of these equations.

(iii) For each solution in step (ii), find r $\frac{\partial^2 z}{\partial x^2}$

s = $\frac{\partial^2 z}{\partial x \partial y}$, t = $\frac{\partial^2 z}{\partial y^2}$

(iv) (a) If $rt\ s^2 > 0$ and $r < 0$ for a particular solution (a,b) of step (ii),then z has a maximum value at (a,b).

(b)) If $rt\ s^2 > 0$ and $r > 0$ for a particular solution (a,b) of step (ii),then z has a minimum value at (a,b).

(c) If $rt\ s^2 < 0$ for a particular solution (a,b) of step (ii),then z has no extreme value at (a,b)

(d) If $rt\ s^2 = 0$, the case is doubtful and requires further investigation.

ASSIGNMENT

1. Examine the extreme values of $x^2 + y^2 + 6x + 12$
2. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
3. A rectangular box open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.
4. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum.

Differentiation Under Integral SIGN

If a function f(x,α) of the two variables x and α, α being called parameter, be integrated w.r.t. x between limits a and b, $\int_a^b f(x,\alpha)dx$ *is a function of* α. for example,

$$\int_0^{\frac{\pi}{2}} \sin\alpha \, dx = -\left[\frac{\cos\alpha}{\alpha}\right]_0^{\pi/2} = -\frac{1}{\alpha}\left(\cos\frac{\pi}{2}\alpha - 1\right)$$

$$= \frac{1}{\alpha}\left(1 - cos\frac{\pi}{2}\alpha\right)$$

$$thus\ in\ general\ \int_a^b f(x,\alpha)dx = F(\alpha)$$

$\boldsymbol{Leibnitz's\ \ Rule}$

$If\ f(x,\alpha) and\ \frac{\partial}{\partial x}\ [f(x,\alpha)] be\ continous\ functions\ of\ x\ and\ \alpha,$ then $\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)dx\right]$

$= \int_a^b \frac{\partial}{\partial x}\ [f(x,\alpha)]\ dx\ where\ a\ and\ b\ are\ constants\ independent\ of\ \alpha.$

$Example\ 1. Evaluate\ \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\ dx\ (a \geq 0) by\ applying\ differentiation\ under\ the\ Integral\ sign.$

$Example\ 2. evaluate\ \int_0^a \frac{\log(1+ax)}{1+x^2}\ dx\ and\ hence\ show\ that\ \int_0^1 \frac{\log(1+x)}{1+x^2}\ dx$
$= \frac{\pi}{8}\log 2$