MATRICES AND ITS APPLICATIONS

- Elementary transformations and elementary matrices
- Inverse using elementary transformations
- Rank of a matrix
- Normal form of a matrix
- Linear dependence and independence of vectors
- Consistency of linear system of equations
- Linear and orthogonal transformations
- Eigen values and eigen vectors
- Properties of eigen values
- Cayley-Hamilton theorem and its applications
- Diagonalization of Matrices, similar Matrices, Quadratic form.

Definition -A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets[ ] is called an m by n matrix ; which is written as m*n matrix. A matrix is also denoted by a single capital letter.

Thus

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

is a matrix of order m * n. It has m rows and n columns. Each of the m * n numbers is called an element of the matrix.

The matrix A is denoted by $[a_{ij}]$

for eg.

$$\mathbf{A} = \begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 3 & 9 & 2 \\ 6 & 0 & 7 \end{bmatrix}.$$

SPECIAL MATRICES

1. Row and column matrices- A matrix having a single row is called a row matrix e.g.,

A matrix having a single column is called a column matrix e.g.,

$\begin{bmatrix} 2 & 5 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$ are raw matrix & column matrix.

2. Square Matrix - A matrix having n rows and n columns is called a square matrix of order n.

For e.g., $\begin{bmatrix} 2 & 4 & 5 \\ 3 & 1 & 7 \\ 4 & 5 & 6 \end{bmatrix}$

The diagonal of this matrix containing the elements 2,1,6 is called the leading or principal diagonal.

A square matrix is said to be **singular** if its determinant is zero otherwise **non-singular**.,

3. Diagonal Matrix – A square matrix all of whose elements except those in the leading diagonal , are zero is called a diagonal matrix.

A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix . For e.g.,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

4. Unit Matrix – A diagonal matrix of order n which has unity for all its diagonal elements , is called a unit matrix or an identity matrix of order n and is denoted by I_n. For e.g., unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Null matrix – If all the elements of a matrix are zeros, it is called a null matrix or zero matrix and is denoted by `**0**' e.g.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6. Symmetric and skew symmetric matrices – A square matrix $A = [a_{ij}]$ is said to be symmetric when for all I and j.

$$a_{ij} = a_{ji}$$

If $a_{ij} = - a_{ji}$ for all I and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix. Examples of symmetric and skew-symmetric matrices are respectively.

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

7. Triangular Matrix – A square matrix all of whose elements below the leading diagonal are zero, is called upper triangular matrix. A square matrix all of whose elements above the leading diagonal are zero, is called a lower triangular matrix. Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ -5 & 3 & 0 \\ 4 & 6 & 2 \end{bmatrix}$$

are called upper triangular matrix & lower triangular matrix.

MINOR & COFACTORS

- If A is a square matrix, then the **minor** of the entry in the *i*-th row and *j*-th column (also called the ***(i,j)*** **minor**, or a **first minor**) is the deteminant of the subm,atrix formed by deleting the i-th row and j-th column. This number is often denoted $M_{i,j}$.
- The (I,j) **cofactor** is obtained by multiplying the minor by denoted by $C_{i,j}$.
- For ex. consider the 3*3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

- $M_{2,3}$ = det $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$ = det $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ 1-6 = -5

So cofactor of (2,3) entries is: $C_{2,3} = (-)^{2+3} M_{2,3}$

LINK FOR REFERENCE

- Matrix **Algebra:**

https://www.youtube.com

Elementary Transformations(Operations)

(i) Interchange of two rows & two columns

The interchange of ith & jth rows is denoted by R_{ij}

The interchange of ith & jth column is denoted by c_{ij}

(ii) Multiplication of (each element of) a row or column by a non zero number k.

The multiplication of ith row by k is denoted by kR_i

The multiplication of jth column by k is denoted by kc_i

(iii) Addition of k times the elements of a row(or column) to the corresponding elements of another row (or column) , k $\neq$ 0

The addition of k times the jth row to the ith row is denoted by $R_i + kR_j$

The addition of k times the jth row to the ith row is denoted by $C_i + kC_j$

If a matrix B is obtained from a matrix A by one or more E-operation then B is said to equivalent to A . Denoted by A ~ B

Elementary Matrices, Inverse of a matrix by Gauss Jordan method

- The matrix obtained from a unit matrix I by applying it to any one of the E-operations(elementary operations) is called an elementary matrix. A^{-1}
- **Gauss Jordan method:**
- The elementary row(not column) operations which reduce a square matrix A to the unit matrix , give the inverse matrix
- Working rule: To find the inverse of A by E-row operations , we write A and I side by side and the same operations are performed on both . As soon as A is reduced to I , I will reduce to A^{-1}

ASSIGNMENT II(A)

Use Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

NPTEL LINK FOR REFERENCE

Inverse of matrix	http://nptel.ac.in/courses/111105035/3

RANK OF A MATRIX

- **A matrix is said to be of rank r when**

(i) it has atleast one non-zero minor of order r,

(ii) every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

(i) If a matrix has a non-zero minor of order r,its rank is $\geq$ r.

(ii) If all minors of a matrix of order r + 1 are zero,its rank is $\leq$ r.

The rank of a marix A is denoted by $\rho(A)$.

Important points to be remember

- If A is a null matrix then $\rho(A) = 0$
- If A is a not a null matrix then $\rho(A) \geq 1$
- If a is a non singular n x n matrix then $\rho(A) = n$
- if I_n is the n x n unit matrix then hence $\rho(I_n) = n$
- if A is the m x n matrix then $\rho(A) = \min(m, n)$
- If all minors of order r are equal to zero then

  $\rho(A) < r$

DIFFERENT METHODS TO FIND RANK OF A MATRIX

(a) Start with the highest order minor of A. let its order be r . If any one of them is non zero then $\rho(A) = r$.

if all of them zero then taking its next lower order (r-1) and so on till we get a non zero minor then the order of that minor is the rank of A.

* Lots of computational work required * so we adopt second method.

(b) If A is an m x n matrix and by a series of elementary(row or column or both) operations , it can be put into one of the following forms : called **Normal forms**

NORMAL FORM OF A MATRIX

Normal forms

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \begin{bmatrix} I_r & 0 \end{bmatrix} \quad \begin{bmatrix} I_r \end{bmatrix}$$

Where I_r is the unit matrix of order r. hence **ρ(A) = r**

- * elementary operations does not effect the rank of a matrix*

Assignment II (b)

- Determine the rank of the matrix by reducing it to the normal form

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

FOR AN MATRIX A M X N OF RANK R, TO FIND SQUARE MATRICES P & Q OF ORDERS M & N RESPECTIVELY , SUCH THAT PAQ IS IN THE NORMAL FORM $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

- **Working rule:** write A = I A I
- Reduce the matrix on L.H.S.to normal form by applying elementary row or column operation.

Remember

* if row operation is applied on L.H.S. then this operation is applied on pre-factor of A on R.H.S

* if column operation is applied on L.H.S. then this operation is applied on post-factor of A on R.H.S

**** The matrices P & Q are not unique. ****

ASSIGNMENT II(C)

Find non-singular matrices P and Q such that PAQ is in the normal form for the matrix

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$$

LINKS FOR REFERENCE

- Rank of matrix
-
- http://nptel.ac.in/courses/111105035/3
- https://www.youtube.com/watch?v=GU-pJMxABQ0

Solution of a System of Linear Equations

Let the system of equations

(3 equations in 3 unknowns)

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

In matrix notation these equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$or \quad A \quad X = B$$

where A : co- efficient matrix

X : column matrix of unknown

B: column matrix of constant

Types of Equations: Homogeneous & Non Homogeneous

- **case I** : If $d_1 = d_2 = d_3 = 0$ then B = 0 and AX = B reduce to AX = 0 then a system of equations is called a system of Homogeneous linear equations.
- **Case 2** : if at least one of d_1, d_2, d_3 is non zero means $B \neq 0$ then a system of equations is called a system of Non Homogeneous linear equations

There are three possibilities :

(i) The equations have no solution , called inconsistent system of equations

(ii) The equations have one solution(unique solution) ,called consistent system of equations

(iii) The equations have more solutions(infinite solution), called consistent system of equations

(A) FOR A SYSTEM OF NON –HOMOGENEOUS LINEAR EQUATION AX = B

(i) If $\rho[A:B] \neq \rho(A)$, the system is inconsistent

(ii) If $[A:B] = \rho(A) =$ number of unknowns, the system has a unique solution

(iii) If $[A:B] = \rho(A) <$ number of unknowns, the system has un infinite number of solution

(b) For a system of Homogeneous linear equation Ax = 0

(i) X = 0 is always a solution , called null solution or the trivial solution. Thus a homogeneous system is always consistent .

(ii) If $\rho(A)$ = number of unknowns, the system has only the trivial solution.

(ii) If $\rho(A) <$ number of unknowns, the system has an infinite no of non trivial solutions.

WORKING RULE:

(i) The given equation AX = B , find A & B

(ii) Write the augmented matrix $[A : B]$

(iii) By E-row operation on A and B , reduce A to a diagonal matrix , thus getting

$$[A : B] = \begin{bmatrix} p_1 & 0 & 0 & : & q_1 \\ 0 & p_2 & 0 & : & q_2 \\ 0 & 0 & p_3 & : & q_3 \end{bmatrix}$$

Then $p_1 x = q_1 \quad p_2 y = q_2 \quad p_3 z = q_3$

Assignment II(d)

Test for consistency and solve the equations

$$x + 2y + z = 3$$

$$2x + 3y + 2z = 5$$

$$3x - 5y + 5z = 2$$

$$3x + 9y - z = 4$$

Assignment II(e)

For what values of a and b do the equations

$x + 2y + 3z = 6$, $x + 3y + 5z = 9$, $2x + 5y + az = b$ have

(i) unique solution

(ii) no solution

(iii) more than one solution

LINKS FOR REFERENCE

- System of simultaneous Linear equation.
- http://nptel.ac.in/courses/111105035/5
-

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VECTOR

- Any quantity having n-components is called a vector of order n.Therefore the coefficients in a linear equation or the elements in a row or column will form a vector.Thus any n numbers $x_1,x_2,x_3,\ldots\ldots\ldots\ldots x_n$ written in a particular order, constitute a vector x.
- Any ordered n-tuple of numbers is called an n-vector.
- Let $X=(x_1,x_2,\ldots\ldots\ldots x_n)$, $Y=(y_1,y_2,\ldots\ldots\ldots\ldots y_n)$
- **Two operations for vectors:**
- Addition : $X+Y=(x_1+y_{1,}x_2+y_2,\ldots\ldots\ldots\ldots\ldots x_n+y_n)$
- Scalar multiplication $KX==(kx_1,kx_2,\ldots\ldots\ldots kx_n)$

Types of Vectors

Any quantity having n-components is called a vector of order n.Therefore the coefficients in a linear equation or the elements in a row or column will form a vector.Thus any n numbers $x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots x_n$ written in a particular order, constitute a vector x.

Linear dependence. The vectors $x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots x_n$ are said to be linearly dependent, if there exists r numbers

$\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_r$ not all zero, such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots\ldots\ldots\ldots\ldots \lambda_r x_r = 0$$

If no such numbers other than zero, exist, th

vectors are said to be **linearly independent**.

Checking for the Linear dependency & Linear independency of vectors

- **Working rule**: suppose there are any n vectors $x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots x_n$

(i) Consider the relation

$$k_1\, x_1 + k_2\, x_2 + k_3\, x_3 + \cdots \ldots \ldots .\, k_n\, x_n = 0$$

(ii) substituting the values of $x_1, x_2, x_3 \ldots \ldots \ldots . x_n$ and form the equations.

(iii) Now equating corresponding components on both side & we get a **homogeneous system** of n linear equations in m unknowns.

(iv) Form the coefficient matrix

(v) Find the rank of above matrix

THERE ARE TWO POSSIBLE CASES

(i) If **rank of matrix is less than the number of unknown** , the homogeneous system has infinitely many non – zero solution. Thus there exist scalars $k_1, k_2, k_3 \dots \dots \dots . k_n$, nor all zero , such that

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots \dots \dots . k_n x_n = 0$$

The vectors $x_1, x_2, x_3 \dots \dots \dots . x_n$ are **linearly dependent**

(ii) If **rank of matrix is equal than the number of unknown** , the homogeneous system has trivial solution or zero solution. Thus there exist scalars $k_1, k_2, k_3 \dots \dots \dots . k_n$

all are zero, such that $k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots \dots \dots . k_n x_n = 0$

The vectors $x_1, x_2, x_3 \dots \dots \dots . x_n$ are **linearly independent**

Assignment II(f)

Are the vectors linearly dependent? If so, find a relation between them.

Given the vectors

$x_1 = (2,3,1,-1)$, $x_2 = (2,3,1,-2)$, $x_3 = (4,6,2,1)$

NPTEL LINKS FOR REFERENCE

- Linear dependent & Independent vectors
- http://nptel.ac.in/courses/111105035/2

Linear Transformations

Let (x,y) be the co-ordinates of a point P.

In general, the relation Y = AX where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

give linear transformation from n variables

$x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ to the variables $y_1, y_2, y_3, \ldots\ldots\ldots Y_n$
i.e. the transformation of the vector X to the vector Y.

If the transformation matrix A is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation Y = AX we can also write the inverse transformation X = A^{-1} Y. A non-singular transformation is also called a regular transformation.

ASSIGNMENT II(G)

- Represent each of the transformations

  $x_1 = 3y_1 + 2y_2$, $x_2 = - y_1 + 4y_2$, $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$ by the use of matrices and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2

ORTHOGONAL TRANSFORMATION

The linear transformation Y = AX is said to be **orthogonal** if it transforms $y_1^2 + y_2^2 + \ldots\ldots\ldots + y_n^2$ into $x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots + x_n^2$

The matrix A of this transformation is called an **orthogonal matrix**.

Now X'X = $\begin{bmatrix} x_1 & x_2 & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$

= $x_1^2 + x_2^2 + x_n^2 \ldots\ldots\ldots +$

and similarly Y'Y = $y_1^2 + y_2^2 + \ldots\ldots\ldots\ldots + y_n^2$

If Y = AX is an **orthogonal transformation**, then

X'X = $x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots + x_n^2 = y_1^2 + y_2^2 + \ldots\ldots\ldots\ldots\ldots + y_n^2$

= Y'Y = (AX)'(AX) = (X'A')(AX) = X'(A'A)X

ORTHOGONAL MATRIX

- which holds only when $A'A = I$ or when $A'A = A^{-1}A$ or when $A' = A^{-1}$
- Hence a square matrix A is said to be orthogonal if $A'A = AA' = I$
- Also, for an **orthogonal matrix** A, $A' = A^{-1}$

Assignment II (h)

- Verify the matrix is orthogonal

$$\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

NPTEL LINKS FOR REFERENCE

- Linear transformation , Matrix presentation.
-
- http://nptel.ac.in/courses/111105035/4
-

Characteristic Equation

If A is a square matrix of order n, we can form the matrix

A - λ I, where λ is a scalar and I is the unit matrix of order n.

The determent of this matrix equated to zero, i.e.,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{2n} \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix} = 0$$

is called the **characteristic equation** of A.

The roots of this equation are called the **characteristic roots** or **latent roots** or **Eigen values** of A

EIGEN VECTORS

Consider the linear transformation $Y = AX$ (1)

which transforms the column vector X into the column vector Y.

Let X be such a vector which transforms into λX (λ being a non-zero scalar) by the transformation (1)

Then $Y = \lambda X$ (2)

From (1) and (2)

$AX = \lambda X = 0$ or $AX - \lambda IX = 0$ or $(A - \lambda I)X = 0$ (3)

These equations will have a non – trivial solution only if the co-efficient matrix $A - \lambda I$ is singular i.e., if

$|A - \lambda I| = 0$ (4)

Eigen vector

This is the characteristic equation of the matrix A and has n roots which are the **Eigen values** of A. Corresponding to each root of (4), the homogeneous system (3) has a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

which is called an **Eigen vector** or **latent vector**.

Properties of Eigen values

(a) The eigen values of a square A and its transpose A' are the same.

(b) The sum of the eigen values of a matrix is the sum of the elements on the principal diagonal.

(c) The product of the eigen values of a matrix A is equal to $|A|$

(d) If λ is an eigen value of a non-singular matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

(e) If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also its eigen value.

PROPERTIES OF EIGEN VALUES

(f) If λ_1 λ_2……., λ_n are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots\ldots \lambda_n^m$ (m being a positive integer).

(g) The eigen values of an idempotent matrix are either zero or unity.

ASSIGNMENT II(I)

- Find the Eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

LINKS FOR REFERNCE

Eigen values & Eigen vectors ,Diagonalization of Matrix.	http://nptel.ac.in/courses/111105035/6	https://www.youtube.com/watch?v=If8pYknlxnM

Cayley Hamilton Theorem

Statement : Every square matrix satisfies its characteristic equation.

* **Possible Questions based on Cayley-Hamilton theorem**

1. Find Characteristic Equation of a matrix.
2. Verify Cayley-Hamilton theorem .
3. Find the inverse of a matrix.
4. Find the matrix represented by polynomial of a matrix.