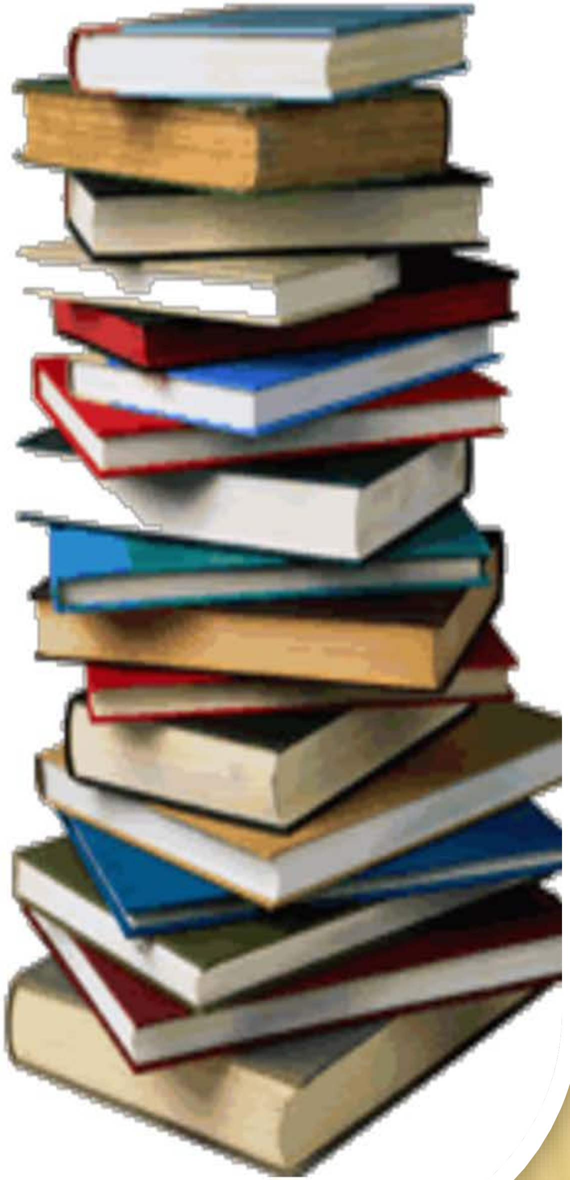
MATHEMATICS-1



SECTION A

Infinite Series

INFINITE SERIES

1. Series: Convergence
2. General Properties
3. Series of positive terms
4. Comparison tests
5. Integral test
6. Comparison of ratio's
7. D' Alembert's ratio test
8. Raabe's test,Logarithmic test
9. Cauchy's root test
10. Alternating series; Leibnitz's rule
11. Absolute Convergence &

    Conditionally convergence

SEQUENCES

A sequence is an ordered progression of numbers, they can be finite or infinite in numbers .

Mathematically,. A Sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. e.g. 1, 5, 9, 13 -----------

Consider the sequence a_1, a_2, a_3, ------------- a_n. This sequence is denoted by $\{a_n\}$

SEQUENCES

A sequence can be defined using a formula to find a_n

$$ex. \quad a_n = 3n - 4$$

$$-1,\ 2,\ 5,\ 8,\ 11,\ \ldots$$

$$ex. \quad a_n = a_{n-1} - 6 \text{ and } a_1 = 14$$

$$14,\ 8,\ 2,\ -4,\ -10,\ \ldots$$

ARITHMETIC SEQUENCES

An Arithmetic sequence is a special type of sequence in which successive terms have a common difference (adding or subtracting the same number each time)

Common difference is denoted ***d***

Formula for arithmetic seq. is:

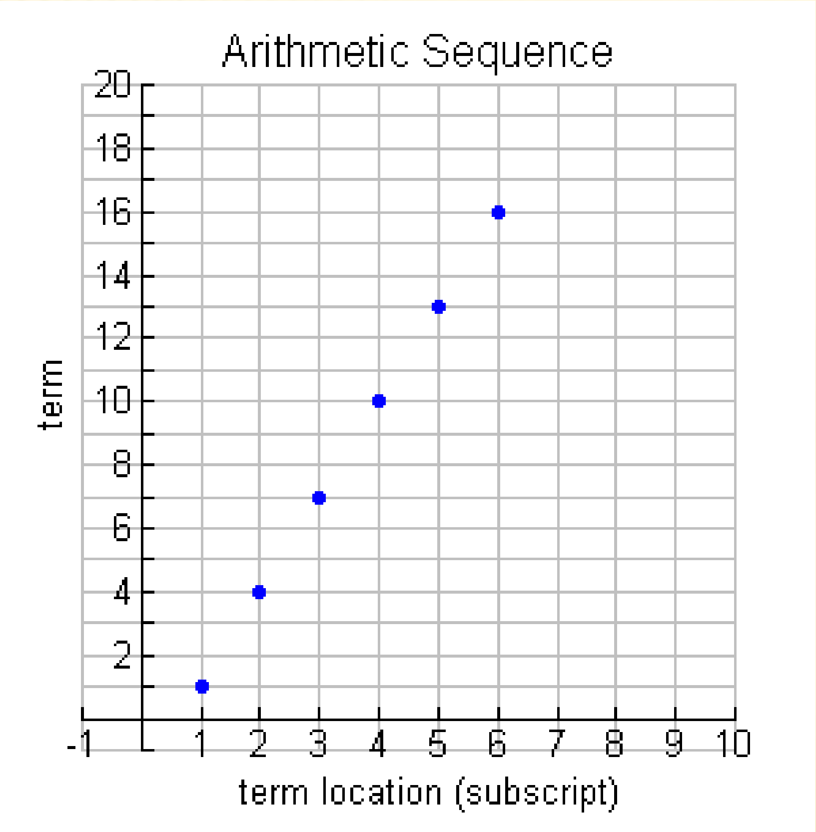
$$a_n = a_1 + (n-1)d$$

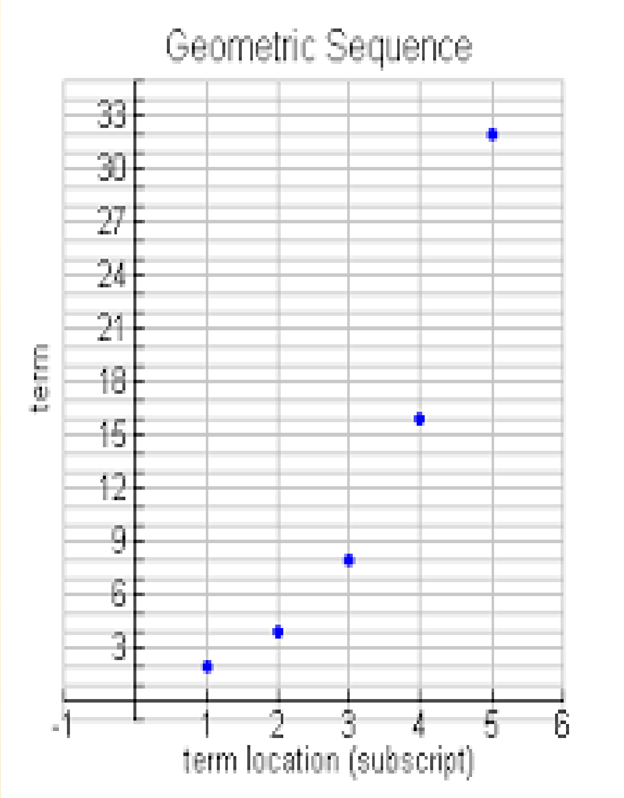
GEOMETRIC SEQUENCES

Geometric sequence is a special type of sequence in which successive term have a common ratio (multiplying or dividing by the same number each time),

the common ratio is denoted *r,f*ormula for geometric seq. is:

$$a_n = a_1 \cdot r^{n-1}$$


Arithmetic Sequence
term
term location (subscript)
20
18
16
14
12
10
8
6
4
2
-1
1
2
3
4
5
6
7
8
9
10


Geometric Sequence
term
term location (subscript)
33
30
27
24
21
18
15
12
9
6
3
-1
1
2
3
4
5
6

Convergence , divergence & oscillatory sequence

(a)If $\lim_{n\to\infty} a_n = a\ finite\ quantity$, then the sequence is said to be convergent

(b) a)If $\lim_{n\to\infty} a_n = +\infty or - \infty$,then the sequence is said to be divergent

(c) If $\lim_{n\to\infty} a_n$ is not unique , then the sequence is said to be oscillatory.

Q. Examine the convergence of the sequence

$$u_n = \frac{3n-1}{1+2n}$$

SERIES

Finite Series:- If u_n is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + ------- u_n$
[i.e. the sum of all terms of the sequence , which are finite in number] is called a **finite Series**.

Infinite Series:- If u_n is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + ------- u_n \pm ----- \infty$
[i.e. the sum of all terms of the sequence , which are infinite in number] is called an **Infinite Series**.

The Infinite Series $u_1 + u_2 + u_3 + --------u_n \pm -----\infty$ is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$.

Series of positive terms:-

If all the terms of the infinite series $\sum u_n = u_1 + u_2 + u_3 + ------u_n \pm -----\infty$ are positive i.e.$u_n > 0 \; \forall \; n$, then the series $\sum u_n$ is called a positive term series.

Partial Sums:-

If $\sum u_n = u_1 + u_2 + u_3 + --------u_n \pm -----\infty$ is an infinite series where the terms may be +ve or –ve , then $S_n = u_1 + u_2 + u_3 + ------u_n$

Is called the nth partial sum of $\sum u_n$. Thus the nth partial sum of an infinite series is the sum of its first n terms. $S_1, S_2, S_3, -----$are first, second, third ------------- partial sums of the series.

Since $n \in N, \{S_n\}$ is a sequence called the sequence of partial sums of an infinite series $\sum u_n$.

PARTIAL SUM FORMULAS

Arithmetic sequence

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d)$$

Geometric sequence

$$S_n = a_1 \cdot \frac{1-r^n}{1-r}$$

Convergence, Divergence and Oscillation of an infinite series:-

An infinite series $\sum u_n$ converges, diverges and oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates(finitely or infinitely).

The series $\sum u_n$ converges if the sequence $\{S_n\}$ of its partial sums converges.

Or we can say $\sum u_n$ is convergent if $\lim_{n\to\infty} S_n = a\ finite\ quantity$

- The series $\sum u_n$ diverges if the sequence $\{S_n\}$ of its partial sums diverges.
- Or we can say $\sum u_n$ is divergent if $\lim_{n\to\infty} s_n = +\infty or -\infty$
- The series $\sum u_n$ oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.
- The series $\sum u_n$ oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.
- Thus, $\sum u_n$ is oscillates infinitely if $\{S_n\}$ oscillating between$+\infty or -\infty$

Q1 test the nature of the series $1 + 3 + 5 + 7 + - - - - \infty$

Let $S_n = 1 + 3 + 5 + 7 + - - -$ n terms

Here $T_n = 2n - 1$

$$S_n = \sum T_n$$

$$= \sum(2n - 1) = 2\sum n - \sum 1$$

$$= 2\frac{n(n+1)}{2} - n$$

$$= n^2 + n - n = n^2$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 = \infty$$

$\Longrightarrow \{S_n\}$ diverges to $+\infty$

So, the given series diverges to $+\infty$

Q2 $1-\frac{1}{5}+\frac{1}{5^2}-\frac{1}{5^3}+------\infty$

The given series being a G.P. series with common ratio $-\frac{1}{5}<1$

$S_n=1-\frac{1}{5}+\frac{1}{5^2}-\frac{1}{5^3}+------n\ terms$

$S_n=1-\frac{1-\left(-\frac{1}{5}\right)^n}{1-\left(-\frac{1}{5}\right)}=\frac{1-(-1)^n\frac{1}{5^n}}{1+\frac{1}{5}}$

$=\frac{5}{6}\left\{1-(-1)^n\frac{1}{5^n}\right\}$

$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{5}{6}\left\{1-(-1)^n\frac{1}{5^n}\right\}$

$=\frac{5}{6}$ $\quad [\because as\ n\to\infty,\frac{1}{5^n}\to 0]$

$\Longrightarrow \{S_n\}$ converges to $\frac{5}{6}$

So, the given series converges to $\frac{5}{6}$

Q3 $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 - - - - \infty$

Here $S_{n=}$ $7 - 4 - 3 + 7 - 4 - 3 + 7 - 4 - 3 - - - -$ to n terms

=0,7,3 according as the number of terms is 3m, 3m+1, 3m+2 respectively.

$S_{n=\begin{Bmatrix} 0,3m \\ 7,3m+1 \\ 3,3m+2 \end{Bmatrix}}$ number of terms

Clearly S_n does not tend to a unique limit, $\{S_n\}$ oscillates finitely.

$\Longrightarrow$The given series oscillates finitely.

Q4. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots\infty$

Let $u_n = \frac{1}{(2n-1)(2n+1)}$

$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

Putting n = 1,2,3,……… n ,we get

$u_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$

$u_2 = \frac{1}{2}\ (\frac{1}{3} - \frac{1}{5})$

$u_3 = \frac{1}{2}(\frac{1}{5} - \frac{1}{7})$

.

.

.

$u_n = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$

Adding $s_n = \frac{1}{2}(1 - \frac{1}{2n+1}) = \frac{1}{2}$

$\Rightarrow$The sequence$\{s_n\}$ converges to $\frac{1}{2}$

$\Rightarrow$The given series converges to $\frac{1}{2}$.

Assignment

Test the nature of the following series:

(i) $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots \infty$

(ii) $1 / 1.3 + 1/ 3.5 + 1/5.7 + \ldots\ldots\ldots\ldots \infty$

NPTEL LINKS FOR REFERENCE

real numbers	http://nptel.ac.in/courses/122104017/1
sequences	http://nptel.ac.in/courses/122104017/2
sequences	http://nptel.ac.in/courses/122104017/3
Series of numbers.	http://nptel.ac.in/courses/122101003/25
Infinite series partial sum	http://nptel.ac.in/courses/122104017/13

Convergence of G.P. Series :

- The Geometric Series 1+ x + x^2 + x^3 + ……… ∞
- Converges if -1<x< 1,i.e. $|x| < 1$
- Diverges if x ≥ 1
- Oscillates finitely if x = -1
- Oscillates infinitely if x ≤ 1

Proof :- (i) when $|x| < 1$.

Since $|x| < 1$,$x^n \rightarrow 0 \; as \; n \rightarrow \infty$

S_n = 1 + x + x^2 + x^3 +………. + x^{n-1}

$$= \frac{1\,(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$\lim_{n\rightarrow\infty} S_n = \frac{1}{1-x}$ (finite value)

⟹ The sequence $\{S_n\}$ is convergent.

⟹ The given series is convergent.

(ii) when x ≥ 1

Subcase I :- when x = 1

S_n = 1 + 1 + 1 + ………………. To n terms.

S_n = n

$\lim_{n\to\infty} S_n = \infty$

The sequence $\{S_n\}$ diverges to ∞.

⟹ The given series diverges to ∞.

Subcase II :- When x >1, $x^n \to \infty$ as n→ ∞

S_n = 1 + x + x^2 + …………….. to n terms

$S_n = \frac{1(x^n - 1)}{x-1}$

$\lim_{n\to\infty} S_n = \infty$

⟹ The sequence $\{S_n\}$ diverges to ∞.

⟹ The given series diverges to ∞.

(iii) when x = -1

S_n = 1 – 1 + 1 – 1 +……….. to n terms.

= 1 or 0 according as n is odd or even.

$\Longrightarrow \lim_{n\to\infty} S_n$ = 1 or 0

$\Longrightarrow$ The sequence $\{S_n\}$ oscillates finitely.

$\Longrightarrow$ The given series oscillates finitely.

When x < -1

X < -1 or –x > 1

Let r = -x , then r > 1

$\therefore r^n \to \infty$ as n→ ∞.

S_n = 1 + x + x^2 + …………… to n terms.

$= \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1-(-r)}$ $[\because x = -r]$

$= \frac{1-(-r)^n}{1+r} = \frac{1-r^n}{1+r}$ or $\frac{1+r^n}{1+r}$ according as n is even or odd.

$\lim_{n\to\infty} S_n = \frac{1-\infty}{1+r}$ or $\frac{1+\infty}{1+r}$ = $-\infty$ or $+\infty$

$\Longrightarrow$ The sequence $\{S_n\}$ oscillates infinitely.

$\Longrightarrow$ The given series oscillates infinitely.

Q 5 Test the convergence of the following series :

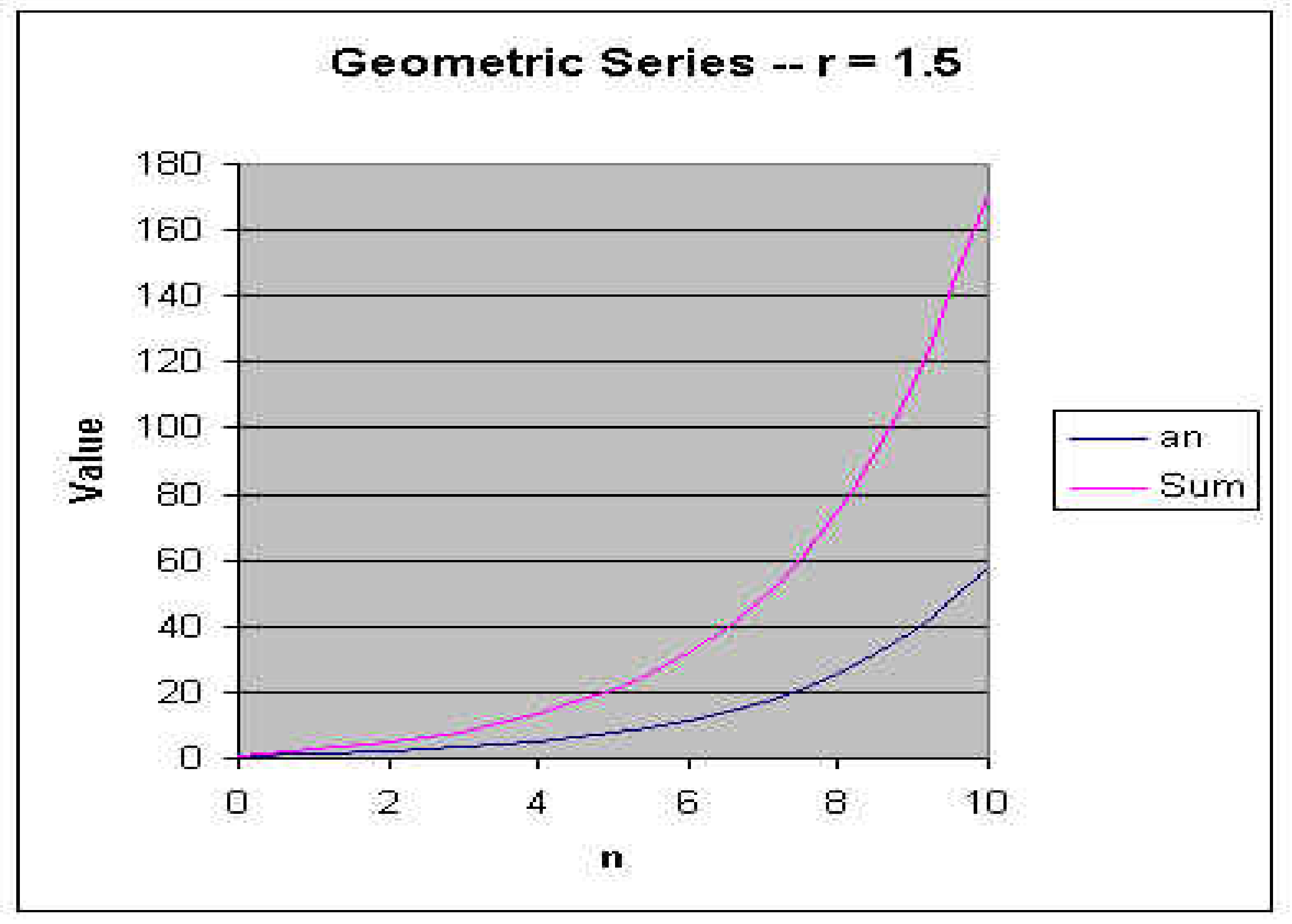
$1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \ldots\ldots\ldots\ldots\ldots\ldots \infty$

$2 + 3 + \frac{9}{2} + \frac{27}{4} + \ldots\ldots\ldots\ldots\ldots\ldots \infty$

$\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

$\frac{3}{4} - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

$1 - \frac{5}{2} + \frac{25}{4} - \frac{125}{8} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$



Geometric Series -- r = 1.5
Value
180
160
140
120
100
80
60
40
20
0
0
2
4
6
8
10
n
an
Sum

Necessary condition for convergence :

If a series $\sum u_n$ is convergent ,then $\lim_{n\to\infty} u_n = 0$

Proof : Let S_n denotes the nth partial sum of the series $\sum u_n$.

The $\sum u_n$ is convergent $\Longrightarrow$ $\{S_n\}$ is convergent.

$\lim_{n\to\infty} S_n$ is finite or unique = s (say)

$\lim_{n\to\infty} S_{n-1}$ = s

Now S_n - $S_{n-1} = u_n$

$\therefore$ $\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} u_n$ = s – s = 0

Hence $\sum u_n$ is convergent $\Longrightarrow$ $\lim_{n\to\infty} u_n = 0$

Converse of the above theorem is not always true, i.e. the nth term may tend to zero as n $\to \infty$ even if the series is not convergent.

e.g. $1+ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots\ldots\ldots + \frac{1}{\sqrt{n}} + \ldots\ldots\ldots\ldots \infty$

$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots\ldots\ldots + \frac{1}{\sqrt{n}}$

$(\because m < n \Longrightarrow \frac{1}{\sqrt{m}} > \frac{1}{\sqrt{n}})$

$= \frac{n}{\sqrt{n}} = \sqrt{n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$

$\Longrightarrow$ The series is divergent, as $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Thus $\lim_{n\to\infty} u_n = 0$ is a necessary condition but not a sufficient condition for convergence of $\sum u_{n.}$

Results :

$\sum u_{n.}$ is convergent $\Longrightarrow$ $\lim_{n\to\infty} u_n = 0$

$\lim_{n\to\infty} u_n = 0 \Longrightarrow \sum u_{n.}$ may or may not be convergent.

$\lim_{n\to\infty} u_n \neq 0 \Longrightarrow \sum u_{n.}$ is not convergent.

A positive term series either converges or diverges to $+\infty$.

Comparison Test :

Let $\sum u_n$ and $\sum v_n$ be two positive term series

If $\lim_{n\to\infty} \frac{u_n}{v_n}$ =l (finite or non zero), then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

$\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges, then $\sum u_n$ also converges.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ and $\sum v_n$ diverges, then $\sum u_n$ also diverges.

Assignment

Examine the convergence of the series

$1 + 1/4^{2/3} + 1/9^{2/3} + 1/16^{2/3} +$ ---------------------------

Convergence of Hyper-Harmonic Series (p-series)

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots\ldots + \frac{1}{n^p} + \ldots\ldots\infty$

Converges if p > 1

Diverges if p ≤ 1

Q 6 Test the convergence of the series

$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \ldots\ldots\infty$

This can be written as

$(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots\ldots + \infty) + (\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \ldots\ldots + \infty)$

$= \sum u_n + \sum v_n$

$\sum u_n$ being a G.P.series with c.r. $= \frac{1}{4} < 1$

$\Rightarrow \sum u_n$ is convergent.

$\sum v_n$ is a G.P.series with c.r. $= \frac{1}{9} < 1$

$\Rightarrow \sum v_n$ is also convergent.

$\Rightarrow \sum u_n + \sum v_n$ is also convergent.

Q 7 Test the convergence of the series

$$\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \ldots\ldots\infty$$

Here $u_n = \frac{1}{\sqrt{n\,(n+1)}}$

$$= \frac{1}{n\sqrt{(1+\frac{1}{n})}}$$

Take $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}}$$

$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}} = 1 \;\left(\because \frac{1}{n} \to 0 \text{ as n} \to \infty\right)$

= finite or non zero

So, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge together.But the series $\sum v_n = \sum \frac{1}{n}$ is divergent as p = 1

Hence, the given series $\sum u_n$ is also divergent.

Q 8 Test the convergence of the series

$\sum(\sqrt{n^4 + 1} - \sqrt{n^4 - 1})$

Here $u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$

Rationalizing, we get

$$u_n = \frac{n^4+1-n^4+1}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{n^2\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

Take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{2}{1+1} = \frac{2}{2} = 1$ (finite and non-zero)

So, by the comparison test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge together.But the series $\sum v_n = \sum \frac{1}{n^2}$ is convergent as p = 2 > 1

Hence, the given series $\sum u_n$ is also convergent.

Q 9 Test the convergence of the series

$$\sum \cot^{-1} n^2$$

Here $u_n = \cot^{-1} n^2$

$$= \tan^{-1}\frac{1}{n^2} \qquad (\because \cot^{-1} x = \tan^{-1}\frac{1}{x})$$

$$= \frac{\tan^{-1}\frac{1}{n^2}}{\frac{1}{n^2}} \cdot \frac{1}{n^2} \quad \text{(Dividing and multiplying by } \frac{1}{n^2}\text{)}$$

Take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{\tan^{-1}\frac{1}{n^2}}{\frac{1}{n^2}} \qquad = \lim_{n\to\infty}\frac{u_n}{v_n} = 1 \qquad (\because \frac{\tan^{-1} x}{x} = 1)$$

(finite and non-zero)

So, by the comparison test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge together.But the series $\sum v_n = \sum \frac{1}{n^2}$ is convergent as p = 2 > 1

Hence, the given series $\sum u_n$ is also convergent.

Assignment

Test the convergence and divergence of the following series

(i) $\sum \cot^{-1} n^2$

(ii) $\sum (2n^3 + 5)/(4n^5 + 1)$

D' Alembert' Ratio Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$,then

$\sum u_n$ is convergent if l > 1

$\sum u_n$ is divergent if l < 1

Test fails if l = 1

Q 9 Test the convergence of the series

$$\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots\ldots\ldots\ldots\ldots \infty$$

The given series can be written as

$$\frac{2.5}{1.5} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots\ldots\ldots\ldots\ldots \infty$$

Here $u_n = \frac{2.5.8.11\ldots\ldots\ldots(3n-1)}{1.5.9.13\ldots\ldots\ldots(4n-3)}$

$=$

$$u_{n+1} = \frac{2.5.8.11\ldots\ldots\ldots(3n-1)(3n+2)}{1.5.9.13\ldots\ldots\ldots(4n-3)(4n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n+1}{3n+2}$$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{4+\frac{1}{n}}{3+\frac{2}{n}} = \frac{4}{3} > 1 \quad (\because \frac{1}{n} \to 0 \text{ as } n \to \infty)$

So by D' Alembert' Ratio Test, the given series $\sum u_n$ is also convergent.

Q 10 Test the convergence of the series

$\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots\ldots\ldots\ldots \infty$

Here $u_n = \frac{x^n}{n+1\sqrt{n+3}}$

$u_{n+1} = \frac{x^{n+1}}{n+2\sqrt{n+4}}$

$\frac{u_n}{u_{n+1}} = \frac{x^n}{n+1\sqrt{n+3}} \text{ x } \frac{n+2\sqrt{n+4}}{x^{n+1}}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} . \frac{\sqrt{1+\frac{4}{n}}}{\sqrt{1+\frac{3}{n}}} \frac{1}{x} = \frac{1}{x}$

So by D' Alembert' Ratio Test, the given series $\sum u_n$ is convergent if $\frac{1}{x} > 1$ i.e. x $<$ 1, and divergent if $\frac{1}{x} < 1$ i.e. x $>$ 1.

Ratio Test fails if $\frac{1}{x} = 1$ $i.e.$ x = 1

For x = 1, $u_n = \frac{1}{n+1\sqrt{n+3}}$

$$= \frac{1}{n^{\frac{3}{2}}(1+\frac{1}{n})\sqrt{1+\frac{3}{n}}}$$

Take $v_n = \frac{1}{n^{\frac{3}{2}}}$

$$\frac{u_n}{v_n} = \frac{1}{(1+\frac{1}{n})\sqrt{1+\frac{3}{n}}} = \quad \lim_{n\to\infty} \frac{u_n}{v_n} = 1$$ (finite and non-zero)

So, by the comparison test, both the series $\sum u_n$ and $\sum v_n$ converge or diverge together.But the series $\sum v_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is convergent as p = $\frac{3}{2}$ > 1

Hence, the given series $\sum u_n$ is also convergent.

Assignment

Test the convergence and divergence of the following series

(i) $\sum n^{1/2} /(n^2 + 1)$

(ii) $\sum (n^2 -1)^{1/2} / (n^3 +1)$

Rabee's Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1)$,then

$\sum u_n$ is convergent if l > 1

$\sum u_n$ is divergent if l < 1

Test fails if l = 1

Q 11 Test the convergence of the series

$$1 + \frac{1}{2} . \frac{x^2}{4} + \frac{1.3.5}{2.4.6} \frac{x^4}{8} + \frac{1.3.5.7}{2.4.6.8.10} \frac{x^6}{12} + \ldots\ldots\ldots\ldots\ldots \infty$$

After neglecting first term

$$u_n = \frac{1.3.5.7\ldots\ldots(4n-3)}{2.4.6.8.10\ldots.(4n-2)} . \frac{x^{2n}}{4n}$$

$$u_{n+1} = \frac{1.3.5.7\ldots\ldots(4n-3)(4n-1)(4n+1)}{2.4.6.8.10\ldots.(4n-2)4n(4n+2)} . \frac{x^{2n+2}}{4n+4}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n(4n+2)}{(4n-1)(4n+1)} . \frac{4n+4}{4n} . \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(4n+2)(4n+4)}{(4n-1)(4n+1)} . \frac{1}{x^2} = \frac{1}{x^2}$$

So by D' Alembert's Ratio Test, the given series $\sum u_n$ is convergent if

$\frac{1}{x^2} > 1$ i.e. $x^2 < 1$, and divergent if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$.

Ratio Test fails if $\frac{1}{x^2} = 1$ $i.e. x^2 = 1$

$$\frac{u_n}{u_{n+1}} = \frac{(4n+2)(4n+4)}{(16n^2-1)}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{16n^2+24n+8-16n^2+1}{(16n^2-1)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n(24n+9)}{(16n^2-1)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{24+\frac{9}{n}}{16-\frac{1}{n^2}} = \frac{24}{16} = \frac{3}{2} > 1$$

So by Raabe's Ratio Test, the given series is convergent.

Hence , the given series is convergent if $x^2 \leq 1$, and divergent if $x^2 > 1$.

Logarithmic Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}}$,then

$\sum u_n$ is convergent if l > 1

$\sum u_n$ is divergent if l < 1

Test fails if l = 1

Q 12 Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots\ldots\ldots\ldots\ldots \infty$$

$$u_n = \frac{n^n x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{n+1!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{n+1!}{n+1^{n+1}x^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{n^n}{n+1^{n+1}} \cdot \frac{1}{x} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{x}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{ex}$$

So by Logarithmic Test, the given series $\sum u_n$ is convergent

if $\frac{1}{ex} > 1$ i.e. x< $\frac{1}{e}$, and divergent if $\frac{1}{ex} < 1$ i.e. x > $\frac{1}{e}$

Logarithmic Test fails if $\frac{1}{ex} = 1\ i.e.\ x = \frac{1}{e}$

For x = $\frac{1}{e}$

$$\frac{u_n}{u_{n+1}} = \frac{e}{(1+\frac{1}{n})^n}$$

$$log\frac{u_n}{u_{n+1}} = \log e - n \log\left(1 + \frac{1}{n}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots \ldots \ldots .\infty\right)$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{3n^3} \ldots \ldots \ldots .\infty$$

$$= \frac{1}{2} < 1$$

By Logarithmic Test, the series is divergent.Hence the given series $\sum u_n$ is convergent if x< $\frac{1}{e}$, and divergent if x $\geq \frac{1}{e}$.

Gauss's Test : If for the positive term series $\sum u_n$, $\frac{u_n}{u_{n+1}}$ can be expanded in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O(\frac{1}{n^2})$$

Then $\sum u_n$ is convergent if $\lambda > 1$ and divergent if $\lambda \leq 1$.

Q 12 Test the convergence of the series

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} x + \frac{1^2 \cdot 3^2 . 5^2}{2^2 \cdot 4^2 . 6^2} x^2 + \ldots\ldots\ldots\ldots\ldots \infty$$

$$u_n = \frac{1^2 \cdot 3^2 . 5^2 \ldots\ldots (2n-1)^2}{2^2 \cdot 4^2 . 6^2 \ldots\ldots\ldots (2n)^2} x^{n-1}$$

$$u_{n+1} = \frac{1^2 \cdot 3^2 . 5^2 \ldots\ldots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 . 6^2 \ldots\ldots\ldots (2n)^2 (2n+2)^2} x^n$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} . \frac{1}{x}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \left(\frac{2 + \frac{2}{n}}{2 + \frac{1}{n}} \right)^2 . \frac{1}{x} = \frac{1}{x}$$

So by Ratio Test, the given series $\sum u_n$ is convergent if $\frac{1}{x} > 1$ i.e. x< 1, and divergent if $\frac{1}{x} < 1$ i.e. x > 1

Ratio Test fails if $\frac{1}{x} = 1 \; i.e. \; x = 1$

For x = 1

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2+8n+4}{4n^2+4n+1}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+8n+4-4n^2-4n-1}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n(4n+3)}{4n^2+4n+1}$$

$$= \lim_{n\to\infty} \frac{n(4n+3)}{4n^2+4n+1}$$

$$= \lim_{n\to\infty} \frac{4+\frac{3}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = 1$$

Raabe's test fail.

Now $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left[1 + (-2)\frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right]$$

$$= \left[1 + \frac{2}{n} - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$= \left[1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]$$

It is of the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + \mathrm{O}\left(\frac{1}{n^2}\right)$$

$\lambda = 1$ By Gauss's test, the series is divergent for $x = 1$.

Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Cauchy's Root Test : If $\sum u_n$ is a positive term series, and if

$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = l$, then

$\sum u_n$ is convergent if $l < 1$

$\sum u_n$ is divergent if $l > 1$

Test fails if l = 1

Q 15 Test the convergence of the series

$$\sum \left(\frac{n\,x}{n+1}\right)^n$$

Here $u_n = \left(\frac{n\,x}{n+1}\right)^n$

$$(u_n)^{\frac{1}{n}} = \frac{x}{1+\frac{1}{n}} = \mathrm{x}$$

So, by Cauchy's Root test, the series is convergent if x < 1 and divergent if x > 1.

For x = 1, $u_n = \left(\frac{n}{n+1}\right)^n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0$

The series is divergent for x = 1.

Hence, the given series is convergent if x < 1 and divergent if x≥ 1.

Assignment

Discuss the convergence of the following series :

(i) $1 + 2x/2! + (3^2x^2)/3! + (4^3x^3)/4! + \ldots\ldots\ldots\ldots$

(ii) $x/1.2 + x^2/3.4 + x^3/5.6 + \ldots\ldots\ldots\ldots$ $(x > 0)$

Cauchy's Integral Test : If for x $\geq$ 1, f(x)is a non-negative, decreasing function of x such that f(n) = u_n for all positive integral value of n, then the series $\sum u_n$ and the integral $\int_1^\infty f(x)\ dx$ converge or diverge together.

Q 16 Show that the series $\sum \frac{1}{n^p}$ converges if p> 1 and diverges if

0< $p \geq 1$

Here $u_n = \frac{1}{n^p}$ = f(x)

$\therefore$ f$(x) = \frac{1}{x^p}$

For x $\geq$ 1, f(x) is +ve and decreasing function of x.

$\therefore$ Cauchy's Integral test is applicable.

Case I : When p$\neq$ 1

$\int_1^\infty f(x)\ dx = \int_1^\infty \frac{1}{x^p}\ dx = \int_1^\infty x^{-p}\ dx$

$$\left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

Subcase I : when p> 1 ⇒ p – 1 > 0, so that

$\int_1^\infty f(x)\ dx$ =

$\frac{1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^\infty = -\frac{1}{p-1}\ [0-1]$

$= \frac{1}{p-1} =$ finite value

⇒ $\int_1^\infty f(x)\ dx$ converges.

⇒ $\sum u_n$ is convergent.

Subcase II : when 0 < p < 1, 1 – p > 0, so that

$\int_1^\infty f(x)\ dx$ =

$\frac{1}{1-p}\left[x^{1-p}\right]_1^\infty = \frac{1}{1-p}\ [\infty - 1]$

$= \infty$

⇒ $\int_1^\infty f(x)\ dx$ diverges.

⇒ $\sum u_n$ is divergent.

Case II: when p = 1, f$(x) = \frac{1}{x}$

$\int_1^\infty f(x)\ dx = \int_1^\infty \frac{1}{x}\ dx = [\log x]_1^\infty = \log \infty - \log 1 = \infty - 0 = \infty$

⇒ $\int_1^\infty f(x)\ dx$ diverges.

⇒ $\sum u_n$ is divergent.

NPTEL LINKS FOR REFERENCE

Tests infinite series	http://nptel.ac.in/courses/122104017/14
Test of convergence	http://nptel.ac.in/courses/122101003/26

Alternating Series

A series in which the terms are alternatively positive or negative is called an **alternating series.**

Leibnitz's rule : An alternating series

$u_1 - u_2 + u_3 - u_4 + \ldots\ldots..$

Converges if (i) each term is numerically less than its proceeding term, and

(ii) $\lim_{n\to\infty} u_n = 0$

If $\lim_{n\to\infty} u_n \neq 0$,the given series is oscillatory.

Assignment

Test for the convergence of the following series :

(i) $2 – 3/2 + 4/3 – 5/4 +$ -------------------------------

(ii) $\sum(-1)^{n-1} . n / (2n -1)$

Series of positive or negative terms

- The series of positive terms and the alternating series are special types of these series with arbitrary signs.
- Def. (1) If the series of arbitrary terms

  $u_1 + u_2 + u_3 + ... + u_n + ...$

  be such that the series $|u_1| + |u_2| + |u_3| + ... + |u_n| + ...$

  is convergent , then the series Σu_n is said to be **absolutely convergent**.

  (2) If $\Sigma|u_n|$ is divergent but Σu_n is convergent, then $\Sigma|u_n|$ is said to be **conditionally convergent**.